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Classification

The Number Tree

Natural Numbers

Counting numbers, viz. {1, 2, 3, 4, ……}

Denoted by N

0 is NOT a Natural number

Whole Numbers

Invention of 0 gave rise to set of Whole Numbers

Set of Natural numbers and also 0 (Zero) viz. {0, 1, 2, 3, 4, ……}

Denoted by W

Integers

Invention of positives and negatives gave rise to set of Integers

Negative and positive Natural numbers and 0 viz. {……, –3, –2, –1, 0, 1, 2, 3, ……}

Remember…

Zero is neither positive nor negative

Thus, set of positive integers will not include 0 and will be {1, 2, 3, ……} i.e. set of Natural numbers

Similarly, set of non-negative integers will include 0 (as it is not negative) and will be {0, 1, 2, 3, ……} i.e. set of Whole numbers

The terms 'positive integers' and 'non-negative integers' are used very often, so be very conversant with them.

Fractions

In addition to an integer part, fractions also have 'part of 1'. E.g. one-half of 1 i.e. $1/2$; one-tenth of 1 i.e. $1/10$; two-thirds of 1 i.e. $2/3$; four-seventh of 1 i.e. $4/7$.

Fractions could also have an Integral part in addition to a 'part of 1'. Fraction like $\frac{15}{4}$ are $3+\frac{3}{4}$ i.e. 3 and three-fourths of 1. And can be written in the mixed-form as $3\frac{3}{4}$.

Insight...

<u>Proper fraction</u> is a fractional number less than 1 i.e. it does not have any integral part and is just a 'part of 1'. In a proper fraction the numerator is less than the denominator. E.g. $\frac{2}{3}, \frac{7}{10}, \frac{5}{8}$

<u>Improper fraction</u> is a fractional number more than 1 i.e. it has an integral part and also a 'part of 1'. In an improper fraction, the numerator is more than the denominator. E.g. $\frac{7}{2}, \frac{5}{4}, \frac{13}{10}$. All these numbers can be written in the mixed-form e.g. $3\frac{1}{2}, 1\frac{1}{4}, 1\frac{3}{10}$

Sometimes, though rarely, you would need to compare (which is greater and which is less) between fractions where the difference between the numerators and the denominator is the same, e.g. which among $\frac{1}{3}, \frac{3}{5}, \frac{7}{9}, \frac{15}{17}$ is greatest or which among $\frac{7}{3}, \frac{17}{13}, \frac{23}{19}$ is the least.

In fractions where difference between numerators and denominators is same......

...... if it is a proper fraction, the fraction increases as numerator increases:

$$\begin{array}{ccccccc} \frac{1}{2} & < & \frac{2}{3} & < & \frac{3}{4} & < & \frac{4}{5} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ 0.5 & & 0.66... & & 0.75 & & 0.80 \end{array}$$

...... if it is a improper fraction, the fraction decreases as numerator increases:

$$\begin{array}{ccccccc} \frac{3}{2} & > & \frac{4}{3} & > & \frac{5}{4} & > & \frac{6}{5} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ 1.5 & & 1.33... & & 1.25 & & 1.2 \end{array}$$

Rational Numbers

All Integers and Fractions together form the set of Rational Numbers and the set of Rational numbers is denoted by Q

By definition, it is the set of all numbers that can be expressed in the form, $\frac{p}{q}$, where p and q are Integers and obviously $q \neq 0$.

Identifying Rational Numbers......

All integers can be expressed in the required form, $\frac{p}{q}$, with q = 1. E.g. 3, 5, –10, are same as $\frac{3}{1}, \frac{5}{1}, \frac{-10}{1}$.

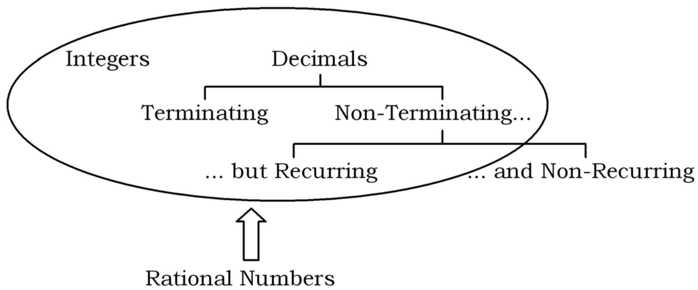
All terminating fractions can be also be expressed in the form, $\frac{p}{q}$, with q being a power of 10 such that the decimal point is eliminated. E.g. 0.3, 4.57, –4.33333 are same as $\frac{3}{10}, \frac{457}{100}, \frac{-433333}{100000}$

...... cont'd

All non-terminating but recurring decimals can also be expressed in the form, $\frac{p}{q}$. E.g.

$0.333... = \frac{1}{3}$; $1.4545... = \frac{16}{11}$; $-3.222...... = \frac{-29}{9}$. Just do the actual division to check that the equality holds. Later we shall see how to find the $\frac{p}{q}$ form of recurring decimals

Thus,



Irrational Numbers

As seen above, decimals that are not rational are those that are non-terminating and non-recurring. This is the set of Irrational Numbers.

Such numbers are $\sqrt{2}, \sqrt{3}, \sqrt[3]{4}$, etc. When found, the numbers do not terminate and nor does any recurring pattern emerge.

Such numbers cannot be converted in the form, $\frac{p}{q}$.

Is π Rational or Irrational?

The constant π is Irrational whereas $\frac{22}{7}$ is Rational (obviously as it is in the p/q form). This should not come as a surprise because the value of π is not exactly $\frac{22}{7}$. The value of π is 3.1415926535...... It is non-terminating and nor does any pattern emerge. $\frac{22}{7}$ is just an approximate value of π.

How on earth did someone come up with such a constant? For a circle of any size, the ratio of the circumference to its diameter is this value.

In fact most of the constants that we use in Math or Physics (*e*, φ, etc) are Irrational.

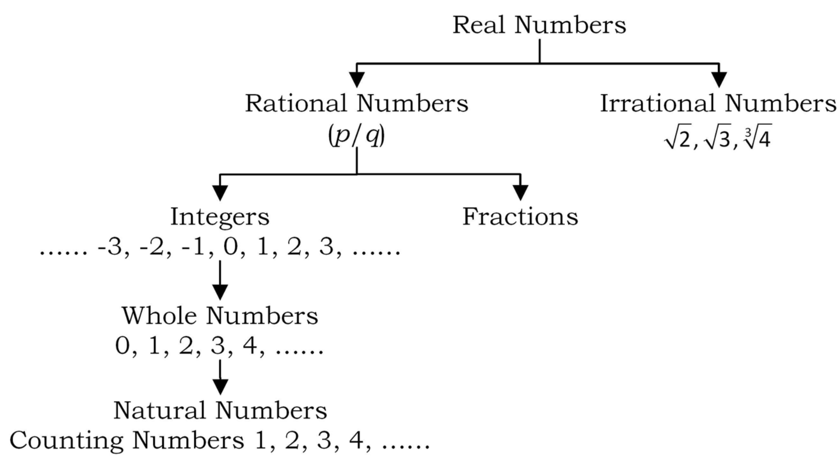
Real Numbers

The set of Rational and Irrational numbers together is the set of Real Numbers and is denoted by R.

Real Numbers are numbers that can be plotted on a number line. Thus even 0.333...... is a unique point on the number and so is $\sqrt{2}$ and other Irrational numbers.

> Can $\sqrt{2}$ be plotted on the real line?
>
> Since $\sqrt{2}$ is a real number and all real numbers can be plotted on a number line, how does one plot $\sqrt{2}$ on the number line? How can one uniquely determine a point corresponding to a non-terminating non-recurring number?

The Number Tree:



> A Teaser......
>
> Which of $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}$ is non-terminating and non-recurring?
>
> One can easily evaluate $\frac{1}{5}, \frac{1}{6}$ and $\frac{1}{8}$ as 0.5, 0.1666... and 0.125 (if one does not already know them). All of them are either terminating or else recurring. Thus, by elimination $\frac{1}{7}$ looks like the answer.
>
> But, it is not so. Once should have realized that each of these is a p/q form and hence have to terminate or else have to recur. Don't believe it? Evaluate $\frac{1}{7}$ and check for yourself that it has to recur. In fact no number of the form p/q will be non-terminating and non-recurring. Thus none of the given fractions is the answer.

Converting Recurring number to a p/q form

We have already seen that non-terminating but recurring numbers are rational numbers i.e. they can be expressed in the p/q form. Here we would study how to convert recurring numbers to the p/q form.

E.g. 1: Convert 0.333...... to a p/q form.

Let $a = 0.3333......$ $\quad \therefore 10a = 3.3333......$

Subtracting, $9a = 3$ $\quad \therefore a = 3/9 = 1/3$ (a p/q form)

E.g. 2: Convert 0.1212...... to a p/q form.

Let $b = 0.1212......$ $\quad \therefore 100b = 12.1212......$

Subtracting, $99b = 12$ $\quad \therefore b = 12/99$ (a p/q form)

Procedure...

We can generalise the above result as follows:

$$\text{Purely recurring number} = \frac{\text{set of digits that recur written once}}{\text{as many 9s as number of digits that recur}}$$

How do we convert numbers of the type 0.1333... or 0.10333... or 0.15757.... i.e. numbers where few digits after the decimal point do not recur and then the set of recurring digits occur? The approach remains identical as follows:

E.g. 3: Convert 0.1333...... to a p/q form

Let $a = 0.1333......$

$\therefore 10a = 1.3333...$ $\quad$ Also, $100a = 13.333...$

Subtracting, $90a = 13 - 1$

$$\therefore a = \frac{13-1}{90} = \frac{12}{90} = \frac{2}{15}$$

E.g. 4: Convert 0.10333...... to a p/q form

Let $b = 0.10333......$

$\therefore 100b = 10.333...$ $\quad$ Also $1000b = 103.333...$

Subtracting, $900b = 103 - 10$

$$\therefore b = \frac{103-10}{900} = \frac{93}{900}$$

E.g. 5: Convert 0.15757...... to a p/q form

Let $c = 0.15757...$

$\therefore\ 10c = 1.5757...$ Also $1000c = 157.5757...$

Subtracting, $990c = 157 - 1$

$$\therefore\ c = \frac{157-1}{990} = \frac{156}{990} = \frac{26}{165}$$

Procedure...

From the above examples, $0.1\overline{3} = \frac{13-1}{90}$; $0.10\overline{3} = \frac{103-10}{900}$; $0.1\overline{57} = \frac{157-1}{990}$. We can generalise as:

Recurring Number =

$$\frac{\text{Non-recurring \& set of reccuring digits written once} - \text{non-recurring digits}}{\text{As many 9s as number of recurring digits, followed by as many 0s as number of non-recurring digits}}$$

Exercise

1. Convert each of the following recurring number to p/q form where p and q are integers.

   i. $0.\overline{123}$ ii. $0.1\overline{23}$ iii. $0.12\overline{3}$ iv. $0.1\overline{02}$ v. $0.10\overline{2}$

   vi. $0.0\overline{102}$ vii. $0.\overline{0102}$ viii. $0.01\overline{02}$ ix. $0.\overline{0120}$ x. $0.00\overline{02}$

2. a & b are two single digit natural numbers such that $0.ababab...... = \frac{8}{11}$. Find the value of $a + b$.

   1. 8 2. 9 3. 10 4. 11 5. 12

3. If $\frac{x}{0.1010......} = \frac{1}{0.222......}$, find the value of x.

   1. 0.444...... 2. 0.555...... 3. 0.4545...... 4. 0.5454...... 5. None of these

4. a and b are two single digit natural numbers such that $0.abab...... \times n$ is an integer value for all values of a and b. What is the least three digit number that n can be?

   1. 990 2. 999 3. 108 4. 198 5. 199

5. If $0.abcabc... = \frac{17}{37}$, find the sum $a + b + c$.

   1. 18 2. 21 3. 24 4. 27 5. None of these

Even and Odd Numbers

Even and Odd are properties of Integers only. For entrance exam purposes, we would limit ourselves to non-negative Integers i.e. 0, 1, 2, 3, ...

Even Number

A number which is divisible by 2 viz. 0, 2, 4, 6, 8, ...

Even numbers are represented by $2n$, where $n = 0, 1, 2,$ ......

Zero is also an Even number.

Odd Number

A number that is not divisible by 2 viz. 1, 3, 5, 7, 9, ...

Odd numbers are represented by $2n + 1$, where $n = 0, 1, 2,$ ......

Is the answer Even or Odd?

Consider odd numbers as $(2n + 1)$ i.e. 1 more than an even number. When odd numbers are added or subtracted one should just consider adding or subtracting these extra ones and check if after the operations would it result in odd or even number of ones. (See figure for better comprehension)

$2n \pm 2m = 2p$

Even ± Even = Even

$(2n + 1) \pm (2m + 1) = 2p$

Odd ± Odd = Even

$2n \pm (2m + 1) = 2p + 1$

Even ± Odd = Odd

$(2n + 1) \pm (2m + 1) \pm (2m + 1) = 2p + 1$

Odd ± Odd ± Odd = Odd

When the operation is multiplication, one should just remember that even numbers are divisible by 2 whereas no odd number can be divided by 2.

$\frac{\cancel{2}n \times a}{\cancel{2}}$

Even × Any number = Even

$\frac{\text{Odd} \times \text{Odd} \times \text{Odd} \times \ldots\ldots}{2}$

Odd × Odd × Odd ...... = Odd

By the above logic we would also have: $\text{Even}^{\text{Any number}} = \text{Even}$ $\text{Odd}^{\text{Any number}} = \text{Odd}$

Exercise

Directions for questions 6 to 15: Fill in the blanks with any one of the following:

1. always even 2. always odd 3. could be even also or odd also

6. $x^3 + x^4$ is ________.

7. $xy^2 + x^2y$ is ________.

8. If $3a + 1$ is even, then a is ________.

9. If $5a - 3$ is odd, then a is ________.

10. If $4a + 2$ is even, then a is ________.

11. If $7a - 4$ is even, then a is ________.

12. If $11a + 10$ is odd, then a is ________.

13. If $10a - 7$ is odd, then a is ________.

14. If $a \times b \times c$ is odd, then $ab + bc + ca$ is ________.

15. $(a - b) \times (b - c) \times (c - a)$ is ________.

Directions for 16 & 17: Choose the correct answer option.

16. Let x, y, and z be distinct integers. x and y are odd and positive and z is even and positive. Which one of the following statements cannot be true?

    1. $(x - z)^2 \times y$ is even
    2. $(x - z) \times y^2$ is odd
    3. $(x - z) \times y$ is odd
    4. $(x - y)^2 \times z$ is even

17. Let x, y and z be distinct integers that are odd and positive. Which of the following statements cannot be true?

    1. $x \times y \times z^2$ is odd
    2. $(x - y)^2 \times z$ is even
    3. $(x + y - z)^2 \times (x + y)$ is even
    4. $(x - y) \times (y + z) \times (x + y - z)$ is odd

Prime and Composite Numbers

Prime and Composite are properties of Natural numbers ONLY.

Prime numbers:

Natural numbers that have exactly two distinct factors viz. 1 and itself

{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, ...}

Composite numbers:

Natural numbers that have more than two distinct factors

Points to note......

1 is neither Prime nor Composite.

2 is the only even Prime number.

There are 25 prime numbers less than 100

Prime numbers greater than 3 are of the form $6n \pm 1$ i.e. one less than a multiple of 6 or one more than a multiple of 6.

Not all numbers of the form $6n \pm 1$ are Prime BUT Prime numbers (> 3) have to be of the form $6n \pm 1$.

Prime numbers are important to us because we can consider all natural numbers to be made up of product of prime numbers. Thus prime numbers can be considered as 'atoms', they are indestructible i.e. they cannot be written down as product of two numbers and also that all other numbers are made by combining many of these 'atoms' together. E.g.

$12 = 2 \times 2 \times 3 = 2^2 \times 3$,

$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5$

$400 = 2 \times 2 \times 2 \times 2 \times 5 \times 5 = 2^4 \times 5^2$

$540 = 2 \times 2 \times 3 \times 3 \times 3 \times 5 = 2^2 \times 3^3 \times 5$

This process of writing any number as a product of prime numbers is called Factorisation and is going to be very useful to us in further problems.

Co-Prime Numbers:

Two numbers are said to be co-prime to each other if they do not have any common factor, other than 1. Thus 8 and 9, though they themselves are not primes, are co-prime to each other. Similarly are 15 and 32.

Also note that......

1 is said to be co-prime to all numbers other than 1.

Exercise

18. State true or false for each of the following:

    i. All prime numbers are odd — True / False

    ii. Product of any two prime numbers could be prime — True / False

    iii. Sum of any two prime numbers is always odd — True / False

    iv. Difference of any two prime numbers is always even — True / False

19. If p is a prime number greater than 3, what is the remainder when

    i. p is divided by 6?

    1. 1   2. 5   3. 1 or 5   4. 2 or 3   5. Cannot be determined

    ii. p^2 is divided by 6?

    1. 1   2. 5   3. 1 or 5   4. 2 or 3   5. Cannot be determined

Directions for questions 20 to 25: Select the correct answer option.

20. In how many ways can 72 be written as a product of two co-prime natural numbers?

    1. 6   2. 5   3. 3   4. 2   5. 1

21. If a, $a + 2$, $a + 4$ are all prime numbers, how many distinct values can a take?

    1. 0   2. 1   3. 2   4. 3   5. More than 3

22. Let p be a prime number greater than 3. Then what is the remainder when $(p^2 + 17)$ is divided by 12?

    1. 3   2. 6   3. 8   4. 9   5. 16

23. If p and q are prime numbers greater than 3 then the greatest number by which $(p^2 - q^2)$ is always divisible is

    1. 12   2. 18   3. 24   4. 30   5. 36

24. How many prime numbers are of the form $n^3 - 1$, where n is any natural number?

    1. 0   2. 1   3. 2   4. 3   5. More than 3

25. How many primes cannot be expressed as a difference of squares of two natural numbers?

    1. 0   2. 1   3. 2   4. 3   5. More than 3

Divisibility Rules

This chapter will help you identify if a number is divisible by another number and if it is not divisible, to identify the remainder. Usually we would need to check for divisibility only by smaller numbers, so we will restrict only to these. While there are hardly any question based on divisibility rules, these rules will help you factorise numbers more easily.

Rule for 3

Sum of digits should be divisible by 3.

If not, the remainder when the number is divided by 3 is same as the remainder when the sum of digits is divided by 3.

E.g. 1: The number 34728 is divisible by 3 because 3 + 4 + 7 + 2 + 8 = 24 is divisible by 3.

The number 13069 is not divisible by 3 because 1 + 3 + 0 + 6 + 9 = 19 is not divisible by 3. When the number 13069 is divided by 3 the remainder will be same as that when 19 is divided by 3 i.e. 1

> **Why does the rule work?**
>
> Any number *abcd* can be written as $1000a + 100b + 10c + d$
> i.e. $999a + 99b + 9c + (a + b + c + d)$.
>
> Each of the numbers other than the bracket is divisible by 3 and hence for the number to be divisible, the quantity in the bracket should be divisible by 3.
>
> While this reasoning is circular in nature (because we are using the fact that 999, 99 are divisible by 3) and there exists a more sound reasoning, for our purpose this reasoning will suffice.
>
> Similar reasoning can be used for rule of 9.

Rule for 4

The two-digit number formed by the last two digits should be divisible by 4.

If not, the remainder when the number is divided by 4 is same as the remainder when the last two digits is divided by 4.

If the last two digits is ‘00’, then also the number will be divisible by 4.

E.g. 2: The number 34728 is divisible by 4 because 28 is divisible by 4

The number 13070 is not divisible by 4 because 70 is not divisible by 4. When the number 13070 is divided by 4, the remainder will be same as that when 70 is divided by 4 i.e. 2

> **Why does the rule work?**
>
> Any number *abcd* can be written as $1000a + 100b + 10c + d$.
>
> Now 100, 1000 and higher powers of 10 are divisible by 4 and thus for the given number to be divisible by 4, $10c + d$ has to be divisible by 4.
>
> Similar reasoning can be used for the rule of 8.

Rule for 6

Check for divisibility by 2 and 3. Only if a number is divisible by both 2 and 3, will it be divisible by 6.

Rule for 8

The last three digits should be divisible by 8.

If not, the remainder when the number is divided by 8 is same as the remainder when the last three digits is divided by 8.

If the last three digits are '000', then also the number will be divisible by 8.

E.g. 3: The number 17243632 is divisible by 8 because 632 is divisible by 8.

The number 1430254 is not divisible by 8 because 254 is not divisible by 8. The remainder when 1430254 is divided by 8 will be same as the remainder when 254 is divided by 8 i.e. 6

Rule for 9

Sum of digits should be divisible by 9.

If not, the remainder when the number is divided by 9 is same as the remainder when the sum of digits is divided by 9.

E.g. 4: The number 14043573 is divisible by 9 because the sum of digits i.e. 27 is divisible by 9.

The number 24736 is not divisible by 9 because the sum of digits is 22 which is not divisible by 9. When the number is divided by 9, the remainder will be same as the remainder when 22 is divided by 9 i.e. 4

Rule for 11

Add all the alternate digits starting with the digit in units place. Let this sum be U. Add all the remaining alternate digits and let this sum be T.

If the difference between U and T is 0 or is divisible by 11, the number is divisible by 11.

If not, the remainder when the number is divided by 11 is U – T. Remember, to find the remainder you have to subtract the sum of the set of digits containing the ten's digit from the sum of the set of digits containing the unit's digit i.e. U – T and NOT T – U.

E.g. 5: Consider the number 39061231. Adding alternate digits starting with the unit digit, we get U = 1 + 2 + 6 + 9 = 18. Adding the other set of alternating digits, we get T = 3 + 1 + 0 + 3 = 7. Since U – T = 18 – 7 = 11, that is divisible by 11, the number 39061231 will also be divisible by 11.

Why does the rule work?

The number *abcd* can be written as $1000a + 100b + 10c + d$, which can be written as $1001a + 99b + 11c + (d - c + b - a)$. Each of the number other than those in the brackets is divisible by 11 and thus for the number to be divisible by 11, the number in the bracket i.e. $(d + b) - (c + a)$ should be divisible by 11.

Again, while this reasoning is circular in nature (as we are using 1001, 99 are divisible by 11) and there exists a more sound reasoning, this logic will suffice for our purpose.

E.g. 6: Consider the number 95341925. Adding alternate digits starting with the unit digit, we get U = 5 + 9 + 4 + 5 = 23. Adding the other set of alternating digits, we get T = 2 + 1 + 3 + 9 = 15. Since U – T = 23 – 15 = 8, which is not divisible by 11, the number 39061231 will also not be divisible by 11. The remainder when the number is divided by 11 will be 8.

Error-Prone Area......

Consider 59439152 and its divisibility with 11.

U = 2 + 1 + 3 + 9 = 15 and T = 5 + 9 + 4 + 5 = 23.

The difference between U and T is 8 and so the number is not divisible by 11.

What will be the remainder? Will it be 8, the difference?

Not really, the remainder when the number is divided by 11 will NOT be 8, but will be 15 – 23 i.e. –8, which we shall later see to be equal to –8 + 11 i.e. 3.

Rules for larger Composite Numbers:

To find if a number is divisible by 6, we check if the number is divisible by 2 and 3. Why are we checking by 2 and 3? Is it because $2 \times 3 = 6$? Do such rules work always? E.g. If a number is divisible by 4 and 6, will it be divisible by 24? Or if a number is divisible by 8 and 9, will it be divisible by 72?

You have to be careful when you frame such rules. The first example is wrong i.e. the number need not be divisible by 24, if it is divisible by 4 and 6. Take the case of 36, it is divisible by 4 and 6 but not by 24. But the second example is correct: if a number is divisible by 8 and 9, it HAS to be divisible by 72.

Since 4 and 6 have a common factor, we cannot say any number divisible by 4 and 6 is divisible by 4×6.

Since 8 and 9 do not have any common factor, if a number is divisible by 8 and 9, it has to be divisible by 8×9.

To form rules of divisibility for higher composite numbers, check for divisibility by two numbers that are co-prime. Thus, to check if a number is divisible by 24, check divisibility by 3 and 8.

Rules for higher Prime Numbers

Any standard text on divisibility rules does not discuss the divisibility rule of 7. It's not because there is no rule. In fact there are atleast three distinct rules to check divisibility by 7. The reason they are not mentioned is that you will hardly ever need to check divisibility by 7. Almost all the situations where-in you would need to check divisibility are covered by the rules mentioned above.

Not just 7, but rules can be framed for the higher prime numbers also. Also there are multiple rules for most of the numbers. Broadly there are two theories for framing divisibility rules for numbers like 7, 13, 17, 19, ... One is based on a 'seed number' and other based on 'oscullators'. Just google for these and learn more if you want to. But be cautioned that you would be wasting your precious time and that there is absolutely no need to know those rules for clearing CAT.

Most national level exams like CAT, XAT, etc do not have stand-alone questions on divisibility rules. But exams like MAT, CET could have questions similar to the following solved examples.

E.g. 7: How many distinct values can x assume if $28357x4$ is divisible by 8?

Since rule for 8 states that the last three digits should be divisible by 8, $7x4$ should be divisible by 8.

Since we cannot divide $7x$ by as we do not know x, lets start with assuming $x = 0$ and then proceed.

For $x = 0$, the last three digits is 704, which is divisible by 8 and thus x can assume the value 0.

Now as we increment x by 1, we keep adding 10 to the number $7x4$. Since 704 is divisible by 8, on adding 10 or 20 or 30 to the number, we will not get a number divisible by 8 (because 10, 20 or 30 are not divisible by 8). Only on adding 40 or 80 to 704 would we get a number divisible by 8. Thus x can assume values of 0, 4 or 8 i.e. 3 values.

E.g. 8: If the number $18601x57y$ is divisible by 72, find the value of $x + y$.

Since the number is divisible by 72, the number will be divisible by 8 and 9. Rather than starting with rule of 9, which will include both the unknowns x and y, you should start with rule of 8 because this will involve only y.

$57y$ should be divisible by 8. Since 57 divided by 8 leaves a remainder of 1, $1y$ should be divisible by 8 i.e. y can be only 6.

Now, $18601x576$ is divisible by 9 and thus the sum of digits should be divisible by i.e. $34 + x$ should be divisible by 9 i.e. x has to be 2.

Thus, $x + y = 2 + 6 = 8$.

E.g. 9: If the number $1735x86y4$ is divisible by 11, what is the least value that $x - y$ can assume?

Sum of digits in alternate places is $4 + 6 + x + 3 + 1$ and $y + 8 + 5 + 7$ i.e. $14 + x$ and $20 + y$.

If now one assumes the difference between the sum as zero and find the answer to be $(14 + x) - (20 + y) = 0$ i.e. $x - y = 6$, one would go wrong. With $x - y = 6$, the number will indeed be divisible by 11, but the question is asking what is the least value of $x - y$.

For $x - y$ to be least, x should be as small as possible and y should be as large as possible i.e. $20 + y$ would be larger than $14 + x$. Thus prompting us to conclude that $(20 + y) - (14 + x) = 0$ or 11 or 22 …

Thus we get $y - x = -6$ or 5 or 16.... But the difference between x and y has to be single digit number as both x and y are digits and thus $y - x = -6$ or 5 are the only valid value. Thus the value of $x - y$ could be either 6 or –5, the least of them being –5.

Exercise

1. The number $94220p31q$ is divisible by 88. What is the value of $p + q$?

   1. 7 2. 9 3. 11 4. 13 5. 15

2. Find the value of x if the number $58215x237$ is divisible by 11?

   1. 9 2. 8 3. 7 4. 6 5. 5

3. How many different values can x take if the number $2506x8$ is divisible by 8?

   1. 0 2. 1 3. 2 4. 3 5. 4

4. If the number $425x36$ is divisible by 72, find the value that x can assume.

   1. 1 2. 3 3. 5 4. 7 5. 9

5. If $8537x54$ is divisible by 3, how many values can x take?

   1. 0 2. 1 3. 2 4. 3 5. 4

6. If $51062x4$ is divisible by 12, how many values can x take?

   1. 0 2. 1 3. 2 4. 3 5. 4

7. When 1000 is added to 459x251 and the resulting number is divided by 11, the remainder is 8. Find x.

   1. 3   2. 5   3. 7   4. 8   5. 9

8. How many possible pairs of values of (x, y) exist such that the number 42xy60 is divisible by 72?

   1. 2   2. 3   3. 4   4. 5   5. 6

9. What is the remainder when the number 5821x59x243 is divided by 11, where x is any single digit whole number?

   1. 3   2. 5   3. 8   4. 10   5. No unique remainder.

10. If the number 3422213xy is divisible by 99, find the values of $x + y$.

    1. 8   2. 9   3. 10   4. 11   5. 12

Puzzle

Form a 10-digit number using each of the digits 0, 1, 2, ..., 9 exactly once such that the number formed by the first two digits (from right end) is divisible by 2; the number formed by the first three digits is divisible by 3; the number formed by the first four digits is divisible by 4; and so on till the number formed by the 10 digits is divisible by 10.

There exists only one such number. The solution would involve a bit of hit and trial as well after all logic is exhausted.

Indices

Rules of Indices:

Rule 1: $a^m \times a^n = a^{m+n}$

Explanation: $(a \times a \times a\ldots_{\ m\text{ times}}) \times (a \times a \times a \ldots_{\ n\text{ times}}) = a \times a \times a \ldots_{\ m+n\text{ times}}$

Rule 2: $a^m \div a^n = a^{m-n}$

Explanation: $\dfrac{a \times a \times a \ldots_{\ m\text{ times}}}{a \times a \times a \ldots_{\ n\text{ times}}} = a \times a \times a \ldots_{\ m-n\text{ times}}$

Rule 3: $\left(a^m\right)^n = a^{m \times n}$

Explanation: $(a \times a \ldots_{\ m\text{ times}}) \times (a \times a \ldots_{\ m\text{ times}})_{\ n\text{ such brackets}} = a \times a \times a \ldots_{\ m\times n\text{ times}}$

> Don't Confuse……
>
> The two expressions $\left(2^3\right)^2$ and 2^{3^2} are different.
>
> In the first expression, we use the above rule and it evaluates to $\left(2^3\right)^2 = 2^6 = 64$
>
> In the second expression, the absence of brackets implies that we will first have to evaluate 3^2. Thus, $2^{3^2} = 2^9 = 512$.
>
> By this logic, $5^{3^{2^2}} = 5^{3^4} = 5^{81}$, whereas $\left(\left(5^3\right)^2\right)^2 = 5^{3\times2\times2} = 5^{12}$

Rule 4: $(a \times b)^n = a^n \times b^n$

Explanation: $(a \times b) \times (a \times b) \times (a \times b) \ldots_{n\text{ times}} = (a \times a \times a \ldots_{\ n\text{ times}}) \times (b \times b \times b \ldots_{\ n\text{ times}})$

Rule 5: $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Explanation: $\left(\dfrac{a}{b}\right) \times \left(\dfrac{a}{b}\right) \times \left(\dfrac{a}{b}\right) \ldots_{\ n\text{ times}} = \dfrac{a \times a \times a \times \ldots\ldots_{n\text{ times}}}{b \times b \times b \times \ldots\ldots_{n\text{ times}}}$

Rule 6: $a^0 = 1$

> How about 0^0?
>
> In different fields of mathematics, based on the context, 0^0 is defined differently. In combinatronics and binomial, it could be taken as 1; in calculus it is taken as 0 or as Undefined. In quite a few cases it is also taken as Indeterminate i.e. it's value cannot be found uniquely.
>
> However we would never come across any case of 0^0 and thus let's forget about it.

Rule 7: $a^{-n} = \frac{1}{a^n}$ or $\frac{1}{a^{-n}} = a^n$

Explanation: $a^n \times a^{-n} = a^{n-n} = a^0 = 1$

This rule will be very effective to convert negative indices to positive one……

$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ and $\frac{1}{3^{-4}} = 3^4 = 81$

Also by the same logic, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

Rule 8: $\sqrt[n]{a} = a^{\frac{1}{n}}$

E.g. 1: Simplify: $\frac{x^3y^{-2}}{x^{-1}y^2} \div \frac{x^{-2}y^{-3}}{y^{-4}}$

$$\frac{x^3y^{-2}}{x^{-1}y^2} \div \frac{x^{-2}y^{-3}}{y^{-4}} = x^{3-(-1)}y^{-2-2} \div x^{-2}y^{-3-(-4)} = \frac{x^4y^{-4}}{x^{-2}y^1} = x^6y^{-5} = \frac{x^6}{y^5}$$

E.g. 2: Simplify $\left(\frac{x^3y^{-2}}{x^{-1}y^2}\right)^2$

$$\left(\frac{x^3y^{-2}}{x^{-1}y^2}\right)^2 = \left(x^{3-(-1)}y^{-2-2}\right)^2 = \left(x^4y^{-4}\right)^2 = x^8y^{-8}$$

E.g. 3: $3^{40} = 27^?$

$$3^{40} = \left(3^3\right)^{\frac{40}{3}} = 27^{\frac{40}{3}}$$

E.g. 4: $16^{20} = 8^?$

$$16^{20} = \left(2^4\right)^{20} = 2^{80} = \left(2^3\right)^{\frac{80}{3}} = 8^{\frac{80}{3}}$$

E.g. 5: Find the value of n if $0.4^n = 6.25^3$

$0.4^n = 6.25^3$

$$\Rightarrow \left(\frac{4}{10}\right)^n = \left(\frac{625}{100}\right)^3 \Rightarrow \left(\frac{2}{5}\right)^n = \left(\frac{25}{4}\right)^3$$

$$\Rightarrow \left(\frac{2}{5}\right)^n = \left(\frac{5}{2}\right)^6 \Rightarrow \left(\frac{2}{5}\right)^n = \left(\frac{2}{5}\right)^{-6}$$

Since the base are equal the two powers would be equal only when the indices are equal i.e. $n = -6$.

Handy Tip in Comparing a^b and b^a ......

If two distinct numbers, a and b, are both greater than 3, then among a^b and b^a, the number having the higher exponent is larger.

Thus, $3^4 > 4^3$ and $17^{19} > 19^{17}$

But remember that both a and b have to be greater than 3. Because if it is not so, we have the following classic cases of exception that are worthwhile to remember......

$3^2 > 2^3$, $2^4 = 4^2$ and obviously $n^1 > 1^n$, where n is any number more than 1.

Exercise

1. $\dfrac{1}{81^{-\frac{3}{4}}} - \dfrac{1}{125^{-\frac{2}{3}}} - \dfrac{1}{64^{-\frac{1}{6}}} =$

   1. 0 2. 1 3. 2 4. 3 5. 5

2. $\sqrt[4]{\sqrt{0.00000256}} =$

   1. 0.0002 2. 0.002 3. 0.02 4. 0.2 5. 2

3. $\left(\dfrac{n^p}{n^q}\right)^{p+q-r} \times \left(\dfrac{n^q}{n^r}\right)^{q+r-p} \times \left(\dfrac{n^r}{n^p}\right)^{r+p-q} =$

   1. n 2. 0 3. 1 4. n^{pqr} 5. $n^{\frac{1}{pqr}}$

4. $\left(\dfrac{n^p}{n^q}\right)^{\frac{1}{pq}} \times \left(\dfrac{n^q}{n^r}\right)^{\frac{1}{qr}} \times \left(\dfrac{n^r}{n^p}\right)^{\frac{1}{pr}} =$

   1. n 2. 0 3. 1 4. n^{pqr} 5. $n^{\frac{1}{pqr}}$

5. What is the value of n in the equality: $32^4 = 16^n$

   1. 2 2. $\dfrac{1}{2}$ 3. 5 4. $\dfrac{8}{5}$ 5. $\dfrac{5}{8}$

6. What is the value of n in the equality: $243^{\frac{1}{3}} = 27^n$

   1. $\dfrac{5}{3}$ 2. $\dfrac{3}{5}$ 3. 9 4. $\dfrac{5}{9}$ 5. $\dfrac{9}{5}$

7. $\left(\frac{3^4}{2^6}\right)^3 \times \left(\frac{4^3}{27^4}\right)^{-1} = \left(\frac{3}{2}\right)^n$. Find n.

1. 0 2. 1 3. –12 4. 24 5. –24

8. $2^{2n-1} = \frac{1}{8^{n-3}}$. Find n.

1. 0 2. 1 3. 2 4. 4 5. 8

9. $\left(\frac{p}{q}\right)^{2n-1} = \left(\frac{q}{p}\right)^{n-8}$. Find n.

1. –7 2. –9 3. 3 4. –3 5. 5

10. $2^{n-1} + 2^{n+1} = 320$. Find n.

1. 5 2. 6 3. 7 4. 8 5. 9

11. If $pqr = 1$, the value of the expression $\frac{1}{1+p+q^{-1}} + \frac{1}{1+q+r^{-1}} + \frac{1}{1+r+p^{-1}}$ is equal to

1. $p + q + r$ 2. $\frac{1}{p+q+r}$ 3. $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$ 4. 1 5. $p^{-1} + q^{-1} + r^{-1}$

12. Which among $2^{\frac{1}{2}}, 3^{\frac{1}{3}}, 4^{\frac{1}{4}}, 6^{\frac{1}{6}}, 12^{\frac{1}{12}}$ is the largest?

1. $2^{\frac{1}{2}}$ 2. $3^{\frac{1}{3}}$ 3. $4^{\frac{1}{4}}$ 4. $6^{\frac{1}{6}}$ 5. $12^{\frac{1}{12}}$

A Puzzle and some home-work ......

If a is a positive real number and $a^m > a^n$, does it imply that $m > n$?

Clue: Draw the graph of the values of a^x v/s x, once with $a = 2$ and once with $a = \frac{1}{2}$. Let x assume all real values and plot it on the horizontal scale i.e. X axis and plot the corresponding values of a^x on the vertical scale i.e. Y axis.

Does comparing the two graphs help you to solve the puzzle?

Also do remember the shape of the graphs.

Cyclicity of unit digit in a power

The concept of cyclicity is used to identify the unit digit of any power, say a^b.

The unit digit in any operation depends on only the unit digit of the numbers used.

Find the unit digit of 357 × 4084.

To find the unit digit, we do not have to perform the complete multiplication. The unit digit will just depend on the unit digits of the two numbers……

$$\begin{array}{rrrr|c|} & & 3 & 5 & 7 \\ \times & 4 & 0 & 8 & 4 \\ \hline & ? & ? & ? & 8 \end{array}$$

Similarly, the unit digit of 629 + 48 × 203 – 25 will be the unit digit of
9 + 8 × 3 – 5 i.e. 9 + 4 – 5 i.e. 8. See the diagram to understand the above.

$$\begin{array}{rrrr|c|} & & 6 & 2 & 9 \\ + & ? & ? & ? & 4 \\ - & & & 2 & 5 \\ \hline & ? & ? & ? & 8 \end{array}$$

We would usually have to find the unit digit of a power e.g. unit digit of 4132^{19} or 27^{51} or so on. As learnt above the unit digit of 4132^{19} or of 27^{51} will be respectively the same as the unit digit of 2^{19} and that of 7^{51}. Thus we have to acquaint ourselves with the unit digit of powers of 1 to 9.

The unit digit of successive powers of any number (2^1, 2^2, 2^3, 2^4, 2^5, …… or 3^1, 3^2, 3^3, 3^4, 3^5, ……) repeat themselves in a particular cycle.

Concept of a cycle & finding any term of a cycle

Finding the unit digit of successive powers of 2……

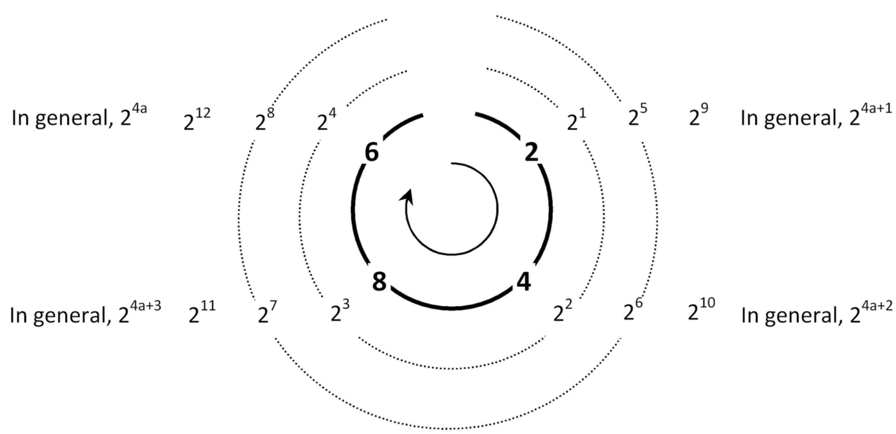
Power of 2	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9
Unit Digit	$\begin{array}{r} \\ \hline 2\end{array}$	$\begin{array}{r} 2 \\ \times\ 2 \\ \hline 4\end{array}$	$\begin{array}{r} 4 \\ \times\ 2 \\ \hline 8\end{array}$	$\begin{array}{r} 8 \\ \times\ 2 \\ \hline *\ 6\end{array}$	$\begin{array}{r} *\ 6 \\ \times\ 2 \\ \hline *\ 2\end{array}$	$\begin{array}{r} *\ 2 \\ \times\ 2 \\ \hline *\ 4\end{array}$	$\begin{array}{r} *\ 4 \\ \times\ 2 \\ \hline *\ 8\end{array}$	$\begin{array}{r} *\ 8 \\ \times\ 2 \\ \hline *\ 6\end{array}$	$\begin{array}{r} *\ 6 \\ \times\ 2 \\ \hline *\ 2\end{array}$

Observing the unit digit (and also the working of finding the unit digit), we realize that the unit digit repeat themselves in a cycle of 2, 4, 8, 6.

> Puzzle......
>
> What is the 72th term in the series *a*, *b*, *c*, *d*, *e*, *a*, *b*, *c*, *d*, *e*, *a*, *b*, *c*, *d*, *e*, *a*, *b*, ...?
>
> The above should be a pretty easy puzzle!
>
> The series consists of 5 terms viz. *a*, *b*, *c*, *d* and *e* that are repeated.
>
> Thus the entire cycle (*a*, *b*, *c*, *d*, *e*) would be completed at the 70th term and the 71st and 72nd terms would respectively be *a* and *b*.
>
> If the above seems simple enough, the same funda will be used here......

Once we know the cycle, any term in the cycle can be found out. If we want to find the 83rd term in the series, 2, 4, 8, 6, 2, 4, 8, 6, 2, 4, ..., we would again go to a multiple of 4 that is just less than 83, in this case 80, and conclude that the 80th term will be the last term of a cycle. Thus the 83rd term will be the third in the cycle i.e. the 83rd term will be 8.

While the above is pretty easy to comprehend, if you are having difficulty, try following the cycle as given in a circular manner in the diagram......



Thus, we see that the cycle ends at every multiple of 4 and then the cycle starts again. To generalize, if the power of 2 is one more than a multiple of 4 i.e. $4a + 1$, then the unit's digit will be 2; if the power is two more than a multiple of 4 i.e. $4a + 2$, then the unit's digit will be 4; and so on.

The cycle of all the digits from 1 to 9 can be found and then the same funda can be used.

Cyclicity of 1, 5 and 6:

Any power of 1, 5 and 6 will end respectively with 1, 5 and 6 (irrespective of the power). Thus the digits 1, 5 and 6 are said to have a cyclicity of 1 and finding the unit digit of their power is not an issue.

Cyclicity of 4 and 9:

The digit 4 and 9 have a cyclicity of 2 i.e. two different values of the unit digit get repeated, as the powers of 4 and 9 are increased successively.

Cycle of unit digit of 4^n as n takes values 1, 2, 3, 4, 5, ... is: 4, 6, 4, 6, 4, ...

Cycle of unit digit of 9^n as n takes values 1, 2, 3, 4, 5, ... is: 9, 1, 9, 1, 9, ...

Thus in the case of 4^n and 9^n one just needs to check if n is odd or even and accordingly the unit digit can be found.

Cyclicity of 2, 3, 7 and 8:

Each of these digits has a cycle of 4 and the respective cycles are...

As n assumes successive natural numbers 1, 2, 3, 4, 5, ...

...... unit digit of 2^n will be 2, 4, 8, 6, 2, 4, 8, 6, 2, 4, ...

...... unit digit of 3^n will be 3, 9, 7, 1, 3, 9, 7, 1, 3, 9, ...

...... unit digit of 7^n will be 7, 9, 3, 1, 7, 9, 3, 1, 7, 9, ...

...... unit digit of 8^n will be 8, 4, 2, 6, 8, 4, 2, 6, 8, 4, ...

Identifying a cycle

You need not memorise all these cycles. You can construct the cycle at the moment you need it. Say you want to find the cycle of the unit digit of 7^n ......

First in the cycle will be 7 itself; next will be the unit digit of 7 × 7 i.e. 9;

next will be the unit digit of 9 × 7 i.e. 3; next will be unit digit of 3 × 7 i.e. 1;

next will be the unit digit of 1 × 7 i.e. 7 and here-on the cycle starts all over again.

Cycle of 8?

First in the cycle will be 8 itself; next will be the unit digit of 8 × 8 i.e. 4;

next will be the unit digit of 4 × 8 i.e. 2; next will be unit digit of 2 × 8 i.e. 6;

next will be the unit digit of 6 × 8 i.e. 8 and here-on the cycle starts all over again.

Once the cyclicity and the cycle is known, any term in the cycle can be easily found by just finding how many extra terms have to be accounted for after complete cycles.

E.g. 1: Find the unit digit of 3^{102}.

We know that the unit digit of successive powers of 3, starting with 3^1, will be the cycle: 3, 9, 7, 1. Thus the cycle will end at each of 3^4, 3^8, 3^{12}, and so on.

The multiple of 4 that is just lower than 102 is 100. Thus 3^{100} will end with 1 and hence unit's digit of 3^{102} will be the second in the cycle i.e. will be 9.

Procedure:

To find unit digit of 2^n or 3^n or 7^n or 8^n

Step 1: Find the remainder when n is divided by 4. Say the remainder is r.

Step 2: The unit digit will be the r^{th} term in the cycle of unit digits. If $r = 0$, the unit digit will be the last term of the cycle, i.e. the cycle just gets completed.

To find the unit digit of 4^n or 9^n

If n is odd, the unit digit of 4^n and 9^n will be the same as unit digit of 4^1 and 9^1 respectively i.e. 4 and 9 respectively.

If n is even, the unit digit of 4^n and 9^n will be the same as unit digit of 4^2 and 9^2 respectively i.e. 6 and 1 respectively.

E.g. 2: Find the unit digit of 8^{75}.

The unit digit of 8^n, as n assumes 1, 2, 3, 4, 5, ... will be the repetition of the cycle: 8, 4, 2, 6.

Thus at 8^{72} the cycle will end and the unit's digit of 8^{75} will be the third term of the cycle i.e. 2.

E.g. 3: What is the unit digit of $357^{59} \times 59^{357}$

Unit digit will be the same as the unit digit of $7^{59} \times 9^{357}$.

7 has a cyclicity of 4. Dividing 59 by 4, we get a remainder of 3. Thus 7^{59} will end with third in the cycle i.e. 3

9 has a cyclicity of 2. Thus 9^{odd} will have unit's digit, same as 9^1 i.e. 9.

Since $3 \times 9 = 27$, unit digit of $357^{59} \times 59^{357}$ will be 7.

E.g. 4: What is the unit digit of $17^{16^{15}}$

The digit 7 has a cyclicity of 4. Thus we need to find the remainder when 16^{15} is divided by 4. But 16^{15} is a multiple of 4 and hence the remainder is 0. Thus, the required unit digit will be the last in the cycle of unit digits of 7^n i.e. last of 7, 9, 3, 1. So the answer is 1.

E.g. 5: What is the unit digit of $19^{17^{15}}$

The digit 9 has a cyclicity of 2. Thus we just have to see if the index is even or odd. 17^{15} is odd. Unit digit of 19^{odd} is 9 (same as unit digit of 9^1)

Cyclicity of last two digits

The last two digits also depict a cycle when the powers are successively increased. However one, the cycle are far too lengthy and second, you would have to remember the cycle for all numbers 01 to 99 (because the last two digits of a power would depend on the last two digits of the base and not just the unit digit).

However there are other ways to identify the last two digits. These methods are given as an appendix after the exercise because they involve prior knowledge of binomial theorem.

Exercise

1. Find the unit digit in each of the following cases:

   i. 423^{423} ii. 413^{7753} iii. $53^{53} \times 33^{33}$

Directions for questions 2 to 10: Choose the correct answer option for each of the following question. In questions where the variable *n* is used, it refers to a natural number.

2. Find the unit's digit of $222^{333} + 333^{222}$.

   1. 1 2. 3 3. 5 4. 7 5. 9

3. Find the unit's digit of $19^{19^{19^{19^{\cdot^{\cdot^{\cdot}}}}}}$

   1. 1 2. 3 3. 5 4. 7 5. 9

4. What is the unit's digit of $17^{18^{19^{20^{\cdot^{\cdot^{\cdot}}}}}}$

   1. 1 2. 3 3. 5 4. 7 5. 9

5. Find the digit in the ten's position of 5×2^{40}

   1. 0 2. 2 3. 4 4. 6 5. 8

6. For how many two digit values of n would 17^n end with 3?

   1. 25 2. 24 3. 23 4. 22 5. 21

7. What is the largest two digit value than n can take such that 88^n and 22^n have the same unit's digit?

   1. 99 2. 98 3. 97 4. 96 5. 95

8. If the unit's digit of 37^n is 3, what is the unit's digit of 73^n?

   1. 1 2. 3 3. 7 4. 9 5. 3 or 7

9. Find the unit's digit of $8^n + 2^n$ if the unit digit of 4^n is not 6.

   1. 0 2. 2 3. 4 4. 6 5. 8

10. How many distinct values can the unit digit of $1^n + 2^n + 3^n + \ldots + 8^n + 9^n$ assume?

   1. 1 2. 2 3. 3 4. 4 5. 5

Appendix: Cyclicity of last two digits

Case 1: When the base ends with a 1 or 9

E.g. 6: Find the two right-most digits of 41^{99}.

$$(41)^{99} = (1+40)^{99} = 1^{99} + {}^{99}C_1 \times 1^{98} \times 40 + {}^{99}C_2 \times 1^{97} \times 40^2 + \ldots\ldots_{\text{higher powers of 40}}.$$

All terms from the third term onwards will have the last two digits 00 because the power of 40 will be 2 or higher. Thus the last two digits will be determined only by the first two terms and will be last two digits of $1 + 99 \times 40$ i.e. $1 + 60$ i.e. 61

The above method can also be used to find the last three digits. One would only have to consider the first three terms.

E.g. 7: Find the two right-most digits of 79^{79}.

$$(79)^{79} = (-1+80)^{79} = (-1)^{79} + {}^{79}C_1 \times (-1)^{78} \times 80 + {}^{79}C_2 \times (-1)^{77} \times 80^2 + \ldots\ldots_{\text{higher powers of 80.}}$$

All terms from the third term onwards will have the last two digits 00 because the power of 80 will be 2 or higher. Also all of them would be adding up to a positive number because we know that 79^{79} is a positive number.

Thus the last two digits will be determined only by the first two terms and will be last two digits of $-1 + 79 \times 80$ i.e. $-1 + 20$ i.e. 19

Case 2: When the base ends with a 3 or 7

E.g. 8: Find the two right-most digit of 113^{56}.

We know that a square of a number with unit digit 3 would end with a 9 and then we can use the method as described in Case 1.

Since the square of 113 is 12769, we can write 113^{56} as 12769^{28}.

Using the funda explained in Case 1,

$$12769^{28} = (-1+12770)^{28} = (-1)^{28} + {}^{28}C_1 \times (-1)^{27} \times 12770 + {}^{28}C_2 \times (-1)^{26} \times 12770^2 + \ldots\ldots_{\text{higher powers of 12770}}$$

Thus the last two digits will be $1 - 28 \times 12770$ + a large positive number with last two digits being 00.

Thus the last two digits will be $01 - 60 = -59$ + large positive number i.e. last two digits will be 41.

If the last step is not understood, look at the following addition....

$$\begin{array}{rccccc} & & & & {\scriptstyle 9} & {\scriptstyle 1} \\ & ? & ? & \cancel{?} & 0 & 0 \\ - & & & & 5 & 9 \\ \hline & ? & ? & ? & 4 & 1 \\ \hline \end{array}$$

One could just do the following: 01 – 60 + 100 = 100 – 59 = 41. (100 in this case is the substitution for a large number with last two digits being 00).

E.g. 9: Find the two right-most digits of 47^{47}.

$$47^{47} = 47 \times 47^{46} = 47 \times 2209^{23}$$

$$2209^{23} = (-1+2210)^{23} = (-1)^{23} + {}^{23}C_1 \times (-1)^{22} \times 2210 + {}^{23}C_2 \times (-1)^{21} \times 2210^2 + \ldots\ldots_{\text{higher powers of 2210}}.$$

The last two digits of 2209^{23} will be –1 + 30 = 29.

Thus the required two right-most digits will be the two rightmost digits of the product 47 × 29 i.e. 63.

Case 3: When the base ends with 5 or an even digit

In such cases the cycle of the last two digits is pretty short and can be identified by direct multiplication.

E.g. 10: Find the two right-most of 84^{42}?

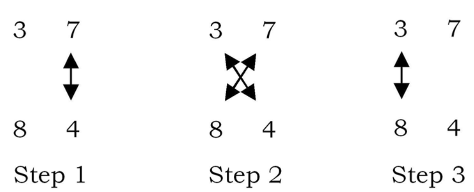
While solving this problem we will have to multiply the last two digits quite often, so first let's learn a short-cut for doing this...

To find the product of two 2-digit numbers......

Say we want to find the product 37 × 84.

Step 1: Multiply the two unit's digit. In this case, 7 × 4 i.e. 28. 8 will be the unit digit of the answer and 2 will be carried forward.

Step 2: Cross-multiply i.e. unit digit of first number with ten's digit of second number and ten's digit of first number with unit's digit of second. See figure. Add the two so-obtained products and any carry-forward from previous step..



Thus 12 + 84 + 2 (carry) = 98. 8 will be the ten's digit of the answer and 9 will be carried forward.

Step 3: Multiply the two ten's digit and add any carry-forward from previous step. In this case, 3 × 8 + 9 = 33. This will be the leading digits.

Thus answer 3388.

In calculations for the task-at-hand, we just need to do till step 2 as we just want the unit and ten's digit.

Focusing only on the two rightmost digits...

Power of 84	84^1	84^2	84^3	84^4	84^5	84^6
Unit Digit	84	84 × 84 ?56	56 × 84 ?04	04 × 84 ?36	36 × 84 ?24	24 × 84 ?16

Power of 84	84^7	84^8	84^9	84^{10}	84^{11}	84^{12}
Unit Digit	16 × 84 ?44	44 × 84 ?96	96 × 84 ?64	64 × 84 ?76	76 × 84 ?84	84 × 84 ?56

After this the cycle will continue. Thus we see that the cycle of the last two digits is: 84, 56, 04, 36, 24, 16, 44, 96, 64, 76 i.e. a cycle of 10 terms. Thus last two digits of 84^{42} will be the 2^{nd} in the cycle i.e. 56.

Tip for a shorter method:

It's best to convert the base into a number with unit digit 6 by taking an appropriate power. Doing this will shorten the cycle......

$84^{42} = 7056^{21}$ and the last two digits would be the same as the last two digits of 56^{21}.

Power of 56	56^1	56^2	56^3	56^4	56^5	56^6
Unit Digit	56	56 × 56 ?36	36 × 56 ?16	16 × 56 ?96	96 × 56 ?76	76 × 56 ?56

Thus the cycle of the last two digit of powers of 56 is: 56, 36, 16, 96, 76 i.e. a cyclicity of 5.

Thus the right-most two digits of 56^{21} will be the first in the cycle i.e. 56. And this will also be the right-most two digits of 84^{42}.

E.g.: 11: Find the two right-most digits of 35^{35}?

Identifying the cycle of the right-most two digits of powers of 35....

Power of 35	35^1	35^2	35^3	35^4	84^5
Unit Digit	35	35 × 35 ?25	25 × 35 ?75	75 × 35 ?25	25 × 35 ?75

Thus the cycle of the last two digits is 35, 25, 75, 25, 75, ...

Thus all odd powers of 35 will end with 75 (except of the first power) and all even powers of 35 will end with 25.

Thus the required two right-most digit of 35^{35} will be 75.

Factorials

$n!$ (read as n factorial) is defined as : $n! = 1 \times 2 \times 3 \times \ldots \times n$

Thus, $1! = 1$ $\quad$ $2! = 1 \times 2 = 2$

$3! = 1 \times 2 \times 3 = 6$ $\quad$ $4! = 3! \times 4 = 24$

> $n! = (n-1)! \times n$
>
> Since, $n! = \underbrace{1 \times 2 \times 3 \times \ldots\ldots \times (n-1)}_{(n-1)!} \times n$, it should be obvious that $n! = (n-1)! \times n$
>
> In words, any factorial = previous factorial × the number.
>
> One should be comfortable in writing any such relation e.g. $n! = (n-2)! \times (n-1) \times n$

$5! = 24 \times 5 = 120$ $\quad$ $6! = 120 \times 6 = 720$

One must remember the above values of 1! to 6!. Also, by definition 0! = 1

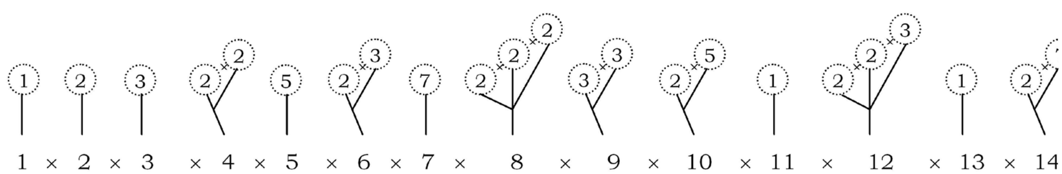
Factorising a Factorial

Let's say we have to factorise 14!.

14! is a large number. Even a 10-digit calculator would be able to accommodate just upto 10!. Thus 14! would be a number in excess of 10 digits. So how does one go about factorising it?

$14! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times \ldots\ldots \times 13 \times 14.$

Each of the numbers on the right hand side can be viewed as product of primes. Thus, 4 can be viewed as 2 × 2; 6 can be viewed as 2 × 3; 18 can be viewed as 2 × 3 × 3; all the primes 2, 3, 5, 7, 11, 13 would remain themselves and cannot be further 'broken'.



Collecting all the 2's together, all the 3's together i.e. all like primes together, we can say that 14! is nothing but a large number formed by the product of one or more of each primes, 2, 3, 5, 7, 11, 13. By visual examination of all the numbers on the RHS, we can easily deduce that primes higher than 13 are not present. Thus,

$14! = 2^a \times 3^b \times 5^c \times 7^d \times 11^e \times 13^f$

In this topic we are going to study the process of identifying the values of a, b, c,……

Highest power of a prime in a factorial

Next let's find *a*, the power of 2 in the factorised form of 14!

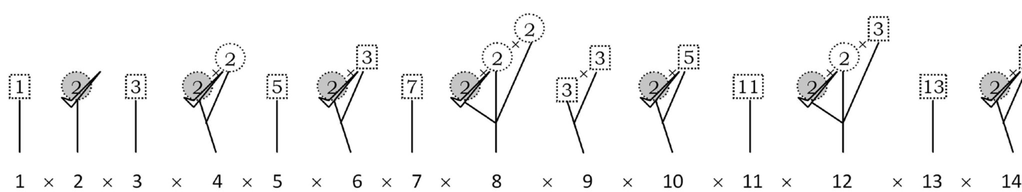
For this we have to collect all possible 2's from each of the numbers 1, 2, 3, 4, ..., 13, 14.

Let's collect all the 2's in 'Rounds'. In one 'round' we will visit each number turn by turn and collect a 2 if it is present in the number. From each number we will collect only one 2 (even if the number has more than one 2's, these extra 2's will be collected in further rounds)

Round 1:

It is obvious that in every multiple of 2 i.e. in 2, 4, 6, 8, ..., 12, 14, there would surely be a 2 present. None of the rest of the numbers i.e. 1, 3, 5, 7, ..., 11, 13 would be contributing a 2.

From 1 to 14, the number of multiples of 2 would be $\frac{14}{2} = 7$. From each of these 7 multiples, one 2 can be collected and thus, we have collected seven 2's.



These seven 2's are the gray-shaded 2's in the above figure. The tick refers to that we have already collected these 2's. Notice that <u>every second number contributes a 2</u>.

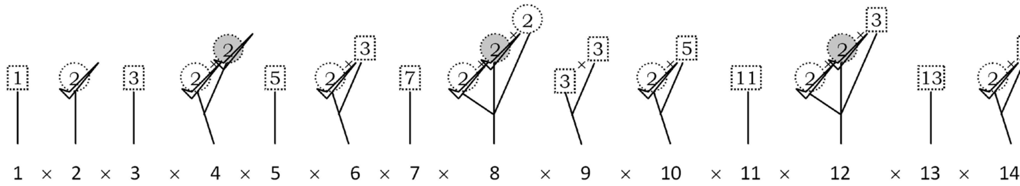
But then these are not all the 2's present. There are few more 2's that are present. And hence we have to revisit each of the number, say round 2, and collect any more 2 that we find, again one from each number.

Round 2:

We will find a further 2 i.e. second 2 only in numbers that originally had two 2's present in them i.e. numbers of the form $2 \times 2 \times n$ i.e. multiples of 4.

From 1 to 14, the number of multiples of 4 would be $\frac{14}{4} = 3$ (we are taking the integral part only as we are interested in the number of multiples of 4). These 4 multiples are 4, 8, 12. And from each of them we collect a 2 i.e. a total of three 2's.

In the following figure these three 2's are the ones in the second row, referring to the second round.

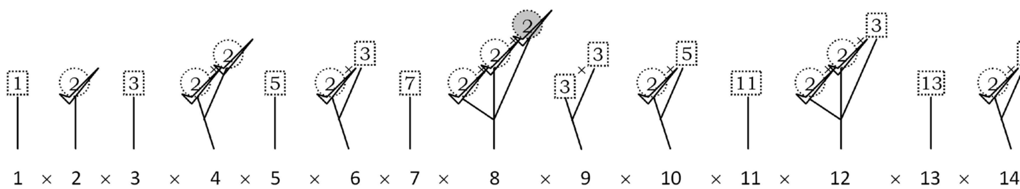


Notice how every 4^{th} number contributes a second 2.

Round 3:

We go through the numbers yet once again to see if any number has any more 2's left. Such numbers would be those that have three 2's in them i.e. of the form $2 \times 2 \times 2 \times n$ i.e. multiples of 8.

From 1 to 14, there would be only 1 multiple of 8 and this number would contribute one more 2.



Numbers that would have a third 2 would have been 8, 16, 24, ... Notice that every 8^{th} number would contribute a third 2. However in our case since there is no 16, 24, ..., we have just one more 2.

Further Rounds:

In this case, we are sure that we have collected all possible 2's and no other 2 remains and so we stop. Else we would have to continue further rounds in the same fashion till we have picked the last available object. In this case, had there been a need for next round, it would involve:

By the logic understood above, every 16^{th} number (i.e. multiples of 16 viz. 16, 32, 48, ... would contribute a fourth 2, but then these numbers are not present in 14! and thus we have collected all the 2's.

Thus, the total number of 2's we collected are 7 + 3 + 1 = 11.

Thus we can say for sure that $14! = 2^{11} \times n$, where n is another natural number which is the product of all the remaining numbers, none of which have a 2 in them.

Question types ......

Once we know the factorised form of a factorial, we should be able to answer most questions based on them. The questions would be camouflaged and not necessarily of the form 'factorise 14!'. Few of the question types are ......

1. Find the highest power of 2 that can divide 14!
2. Find the maximum number of times that 14! can be successively divided by 2.
3. With what least number should 14! be divided so that the quotient is odd i.e. is not a multiple of 2?
4. What least number should be multiplied to 14! so that it is a multiple of 2^{15}?
5. With what least number should 14! be multiplied (or divided) so that the result is a perfect square (or a cube)?

These are just few of the types and there can be many more.

E.g. 1: What is the highest power of 3 in 30!

Number of multiples of 3 from 1 to 30 are $\frac{30}{3} = 10$. Each of these 10 will contribute a 3.

Number of multiples of 3 × 3 i.e. 9 from 1 to 30 are $\frac{30}{9} = 3$. Each of these 3 numbers will contribute one more 3.

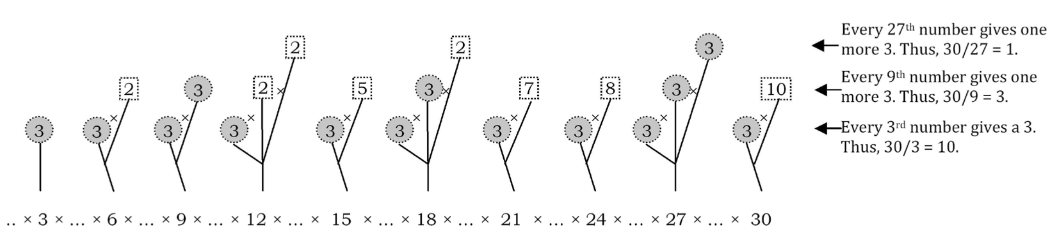
Number of multiples of 3 × 3 × 3 i.e. 27 from 1 to 30 are $\frac{30}{27} = 1$. This one number will contribute a yet another 3.

Thus total number of 3's collected = 10 + 3 + 1 = 14.

Thus, 30! = $3^{14} \times n$, where n is not a multiple of 3.

You should spend a thought about which numbers are the ten numbers that contribute a 3, the three numbers that contribute an additional 3 and the one number that contributes a third 3 as well. See the diagram below if the numbers are not obvious to you.

Since in this example, we are interested in 3's, let's focus only on the 3's



Procedure ......

For the above, there is a shorter procedure to do the same calculation......

Step 1: $\frac{30}{3} = 10$

Step 2: Here we have to calculate $\frac{30}{9}$. But in the earlier step we have already calculated $\frac{30}{3}$. Thus we just need to divide the earlier answer, 10, by 3 i.e. $\frac{10}{3} = 3$.

Step 3: Here we have to calculate $\frac{30}{27}$. But in step 2 we found $\frac{30}{9} = 3$. Thus just divide the earlier answer by 3 i.e. $\frac{3}{3} = 1$.

Thus in short our working is ......

$30 \div 3 = 10$; $10 \div 3 = 3$; $3 \div 3 = 1$

Answer = $10 + 3 + 1$.

Working similarly we can find the highest power of any prime in any factorial. In fact in the case of factorising 14!, as we find for powers of higher prime, the entire process becomes oral.

The power of 5 in 14! would be just 2 because from 1 to 14 we would have just two multiples of 5, each of which will just give one 5.

Similarly the power of 7 will be also be just 2, whereas the power of 11 and 13 will be 1 only.

Thus, $14! = 2^{11} \times 3^5 \times 5^2 \times 7^2 \times 11 \times 13$

Highest power of a composite in a factorial

Consider finding the highest power of 8 in 20!

Erroneous Process

If we use the same process as above, we will just have $\frac{25}{8} = 3$ multiples of 8 from 1 to 25.

Further there would be no multiple $8 \times 8 \times n$, which would contribute any further 8. Thus our answer by the above procedure would be 3. But this will be wrong.

THE ERROR: We would be missing many products which when multiplied would result in a multiple of 8 e.g. 2×4, 6×12, etc. While none of the individual multiplicands are themselves multiples of 8, the product formed would be a multiple of 8.

In such cases we would first need to find the powers of the primes that make up 8, in this case power of 2.

Finding the power of 2 in the factorised form of 25! ......

$25 \div 2 = 12$ $12 \div 2 = 6$ $6 \div 2 = 3$ $3 \div 2 = 1$

Thus the power of 2 in 25! is 12 + 6 + 3 + 1 = 22.

Thus, $25! = 2^{22} \times 3^b \times 5^c \times$ ....

None of the primes 3, 5, 7, ... would help in forming an 8.

8 is formed as 2 × 2 × 2 i.e. three 2's multiply to form one 8. Thus, from twenty-two 2's we have to form groups of three 2's. The number of complete groups formed will be $\frac{22}{3} = 7$.

Thus the highest power of 8 in 25! will be 7.

Alternately using indices, $25! = 2^{22} \times 3^b \times 5^c \times ...... = \left(2^3\right)^7 \times 2 \times 3^b \times 5^c \times ......$

$$25! = 8^7 \times \underbrace{2 \times 3^b \times 5^c \times ......}_{\text{No more 8's can be formed from these}}$$

E.g. 2: Find the highest power of 6 that can divide 50! completely.

In this case since 6 = 2 × 3, we have to find the powers of 2 and 3 in the factorised form of 50!.

Power of 2 will be 25 + 12 + 6 + 3 + 1 = 47

Power of 3 will be 16 + 5 + 1 = 22

Thus we have twenty-two 3's and forty-seven 2's and from this collection we have to form the pair (2 × 3). Thus the maximum number of such pair that we can form is only 22. After this we will run out of 3's even though we have more 2's.

Thus the highest power of 6 that completely divides 50! is 6^{22}.

Short-cut ......

One should have anticipated that the number of 3's would be the limiting factor while forming 6's. How?

1. The number of 3's in 50! will be lesser than the number of 2's because they occur only in every third number whereas 2's occur in every second number.

2. To form a 6 we need to pair one 3 with one 2.

Thus we only needed to find the power of 3 in 50! and that would be our answer.

Don't use this approach indiscrimately, see next example ......

E.g. 3: What is the highest power of 12 that can divide 30!?

$12 = 2^2 \times 3$

Obviously, in 30!, the number of 3's will be less than 2's. But in this example we cannot just find the number of 3's and not worry about the number of 2's. Why?

Because with every 3 we now need two 2's and thus while number of 2's is more than number of 3's, we have to ascertain if they are more than twice or not.

Finding the powers of 2 and 3 in 30! ......

...... powers of 2 in 30! = 15 + 7 + 3 + 1 = 26

...... powers of 3 in 30! = 10 + 3 + 1 = 14

$30! = 2^{26} \times 3^{14} \times n$, where n does not have any 2's or 3's.

We have twenty-six 2's and fourteen 3's. With each 3, we pair up two 2's and it results in one 12. Thus, with fourteen 3's, we need twenty-eight 2's. But we have only twenty-six 2's. Thus in this case, the number of 2's will be in short supply. With twenty-six 2's, we can include thirteen 3's to give us thirteen 12's.

Thus highest power of 12 that can divide 30! is 12^{13}.

Exercise

Questions 1 to 6 are based on basic understanding of factorials.

1. Find the digit in the hundred's position of the product 1! × 2! × 3! × 4! × 5! × 6!.

   1. 0 2. 2 3. 4 4. 6 5. 8

2. Find the remainder when 1! + 2! + 3! + 4! + ... + 151! is divided by 5.

   1. 0 2. 1 3. 2 4. 3 5. 4

3. What is the remainder when $(1!)^3 + (2!)^3 + (3!)^3 + (4!)^3 + \dots + (600!)^3$ is divided by 36?

   1. 4 2. 6 3. 9 4. 12 5. 18

4. Find the smallest number n such that $n!$ is divisible by 990.

   1. 9 2. 10 3. 11 4. 33 5. 99

5. For how many values of n would $(n - 1)!$ not be divisible by n, if n is a natural number less than 20?

   1. 8 2. 9 3. 10 4. 11 5. 12

6. $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots\ldots + 6 \times 6! =$

   1. 5019 2. 5039 3. 5059 4. 5079 5. None of these

Questions 7 to 15 are based on highest power dividing a factorial

7. Find the highest power of 10 that divides 100!.

   1. 10 2. 11 3. 24 4. 97 5. 121

8. What is the highest power of 9 that divides 99! ?

   1. 9 2. 11 3. 12 4. 24 5. 48

9. What is the highest power of 24 that divides 40! ?

   1. 12 2. 13 3. 18 4. 38 5. 56

10. By what least number should 81! be divided such that the quotient is not a multiple of 3?

    1. 3^4 2. 3^9 3. 3^{12} 4. 3^{40} 5. 3^{41}

11. By what least number should 80! be divided such that the quotient is not a multiple of 12?

    1. $2^{78} \times 3^{36}$ 2. $2^{77} \times 3^{36}$ 3. 2^{78} 4. 2^{77} 5. 3^{36}

12. What is the highest power of 6 that can divide 73! – 72! ?

    1. 14 2. 16 3. 34 4. 36 5. 68

13. What is the least number that should be multiplied to 100! to make it perfectly divisible by 3^{50}?

    1. 2 2. 3 3. 6 4. 9 5. 27

14. What is the largest power of 6! that can divide 60! ?

    1. 56 2. 42 3. 28 4. 14 5. 7

15. What is the highest power of 5 that can divide the product of first 100 multiples of 5?

    1. 124 2. 100 3. 97 4. 24 5. 20

The number of zeroes at the end of a product

Few Teasers

1. Consider the product $5 \times 15 \times 25 \times 8$.

While none of the multiplicand is a multiple of 10, yet the product, 15000, has three trailing zeroes. Where have these zeroes come from?

2. Consider the product $10 \times 20 \times 30 \times 40 \times 50$

These are 5 multiples of 10 and thus should have 5 trailing zeroes in the product.

However the product is 12,000,000.

Where has the 6^{th} trailing zero come from?

3. How many trialing zeroes would be there in the product of first 100 prime numbers?

When we are concerned about the number of trailing zeroes in any product, we should look out for 2's and 5's in the multiplicands and not for 10's.

One 2 when multiplied with one 5 would result in one trailing 0

Thus in the product $5 \times 15 \times 25 \times 8$, when factorised, we would find four powers of 5 and three powers of 2. Thus three pairs, 2×5, can be formed and so we have three trailing zeroes.

In the product, $10 \times 20 \times 30 \times 40 \times 50$, each of 10, 20, 30, 40 when factorised provide one and exactly one 5. However when 50 is factorised we have two 5's, $50 = 2 \times 5 \times 5$.

Thus in the product, we have six 5's and we would also have more than six 2's. Thus the final product would have six 0's at the end.

In the product of first 100 prime numbers, if we search for 2's and 5's, we would collect exactly one 2 and one 5 (from the primes 2 and 5 respectively). There would be no further multiple of 2 or 5, else they would not be prime. Thus the product would have exactly one trailing zero.

Conclusion ......

When we need to find the number of trailing zeroes in a product, search for the number of 2's and 5's in the multiplicands.

Since with each 2 we need a 5 to form a 10, the lesser of the number of 2's and 5's will determine the number of zeroes at the end of the product.

Number of zeroes at end of a factorial

A specific product is the case of $n!$, the product of first n natural numbers. Say we have to find the number of zeroes at the end of 75!.

As learnt, we just need to search for 2's and 5's in the product.

Since one 2 when multiplied with one 5 will result in a zero at end, and also since in 75! the number of 5's will be far less than the number of 2's, we just search for number of 5's.

The power of 5 in 75! will be ......

$75 \div 5 = 15$; $15 \div 5 = 3$; ...... $15 + 3 = 18$

Thus $75! = 2^{>18} \times 3^{?} \times 5^{18} \times 7^{?} \times$ ......, and the number of zeroes at the end of 75! will be 18

E.g. 4: Find the number of zeroes at the end of the product $5 \times 10 \times 15 \times 20 \times 25 \times .... \times 150$.

The given expression is
$(5 \times 1) \times (5 \times 2) \times (5 \times 3) \times (5 \times 4) \times (5 \times 5) \times (5 \times 6) \times ...... \times (5 \times 30)$

i.e. $5^{30} \times 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times ... \times 30$ i.e. $5^{30} \times 30!$

30! can further be factorised as $2^{26} \times 5^{7} \times n$, where n has no more 2's or 5's.

Thus the given product is $5^{30} \times 2^{26} \times 5^{7} \times n$ i.e. $2^{26} \times 5^{37} \times n$ and would have 26 zeroes at the end.

Please note that in this example the number of 2's is the limiting factor as in the given product all the numbers used are multiples of 5, and so we have a plethora of 5's.

E.g. 5: Find the number of zeroes at the end of the product of first 60 multiples of 10

Should the answer not be 60?

Since they are 60 multiples of 10, does it imply that there would be exactly 60 trailing zeroes in the product?

Not really, because we have already learnt that to find the number of trailing zeroes we find the number of 2's and 5's in the product. Else we will miss many cases like 50×20 will end NOT with two zeroes but with three zeroes. Thus, our answer has to be more than 60.

The given product = $(10 \times 1) \times (10 \times 2) \times (10 \times 3) \times ...... \times (10 \times 60)$

$= 10^{60} \times 60! = 2^{60} \times 5^{60} \times 60!$

60! will have fewer 5's than 2's and so the number of 5's in the product would be lesser than the number of 2's and would determine the number of trailing zeroes. The power of 5 in factorised form of 60! will be $12 + 2 = 24$

Thus the given product would be of the form $2^{>84} \times 5^{84} \times$ ...... and would have 84 trailing zeroes.

Number of trailing zeroes at end of successive factorials

In this part we would compare the number of trailing zeroes in two successive factorials, say $(n-1)!$ and $n!$.

Since $n! = (n-1)! \times n$, the number of 5's in $n!$ will be equal to the number of 5's in $(n-1)!$ plus the number of 5's present in n.

E.g. Since 20! = 19! × 20, all the 5's that are present in 19! would also be present in 20!. In addition, the number of 5's in the individual number, 20 (= 2 × 2 × 5), would also be included in 20!. Thus 20! would have one more 5 than that contained in 19!.

And consequently the number of trailing zeroes in 20! would be one more than that in 19!.

Next 21! = 20! × 21. Since the number 21 (= 3 × 7) does not have any 5 in it, the number of 5's in 21! and 20! would be the same and they would have the same number of trailing zeroes. In fact the number of zeroes in each of 20!, 21!, 22!, 23! and 24! would be the same as each of them are found by multiplying the earlier factorial with 21 or 22 or 23 or 24, none of which have any 5 present in them.

However, 25! = 24! × 25. And thus the number of 5's (and consequently trailing zeroes) in 25! would be 2 more than the number of 5's in 24!. These extra two 5's have been contributed by the individual number 25 (= 5 × 5).

Again 26!, 27!, 28!, 29! would have the same number of trailing zeroes as 25! because none of 26, 27, 28, 29 contribute any 5.

But 30! would have 1 more trailing zeroes (and not 2 more) than 29!, because the number 30 contributes one (and only one, not two) 5's.

Thus, if we want the number of trailing zeroes in $n!$ and the previous factorial i.e. $(n-1)!$ to differ by three, we just need to make sure that the individual number n, must contribute three 5's. The values that n could take is $5 \times 5 \times 5 \times a$, where a is any natural number. Thus 125! would have 3 more trailing zeroes than 124!, and so would 250! as compared to 249!. There would be infinite such pairs of successive factorials.

E.g. 6: If the number of trailing zeroes in $n!$ is 3 more than the number of trailing zeroes in $(n-1)!$, how many three digit values can n assume?

Solve the question before reading ahead as this example is directly based on the paragraph just above the example.

So that $n!$ has 3 more zeroes than $(n-1)!$, the individual number n must contribute three 5's and thus should be a multiple of 5^3. Thus the possible three digit values of n are 125, 250, 375, 500, 625, 750 and 875 i.e. 7 seven different values (all values from 125 × 1 to 125 × 7).

However 7 is NOT the answer to this question. Among these 7 values of n, there is one value, viz. 625, which is not only a multiple of 5^3 but is also a multiple of 5^4 and thus 625! would have 4 more zeroes than 624!. Thus the number of three digit values that n can take is just 6.

Thus, to be precise, for $n!$ to have 3 more zeroes than $(n-1)!$, the individual number n must contribute EXACTLY three 5's i.e. it should be a multiple of 5^3 but not of 5^4.

The questions need not always be of successive factorials as the following example shows…

E.g. 7: If $n!$ has 20 trailing zeroes and $(4n)!$ has 86 trailing zeores, find the value/s of n.

Since $n!$ has 20 trailing zeroes, the value of n has to be slightly lesser than the 20th multiple of 5 (lesser because there would be multiples of 5 like 25, 50 which would contribute more than one 5). Randomly starting with 80!, we find the number of trailing zeroes to be 16 + 3 = 19. Thus, each of 80! to 84! would have 19 zeroes and each of 85! to 89! would have 20 trailing zeroes. Thus the possible values of n are restricted to 85, 86, 87, 88 and 89.

The possible values of $4n$ are 340, 344, 348, …

The number of trailing zeroes in 340! is 68 + 13 + 2 = 83.

$$\left(\frac{340}{5} = 68, \frac{68}{5} = 13, \frac{13}{5} = 2\right)$$

However we want $(4n)!$ to have 86 zeroes. Thus, going forward from 340! to the other possible values of $4n$, we should include numbers that contribute a total of three more 5's.

344! would also have the same number of trailing zeroes as 340! because none of the extra multiplicands viz, 341, 342, 343 and 344 contribute any 5.

348! would have one more trailing zero i.e. a total of 84 trailing zeroes.

352! would have yet TWO more trailing zeroes than 348! because the individual number, 350, is a multiple of 25 and contributes two more 5's. Thus 352! would have 86 trailing zeroes.

Check: $\frac{352}{5} = 70;\ \frac{70}{5} = 14;\ \frac{14}{5} = 2;\ 70 + 14 + 2 = 86$

And the next possible value of $(4n)!$ i.e. 356! would have 87 trailing zeroes, one more than 352! because the number 355 would contribute one more 5.

Thus the only value of $4n$ is 352 i.e. the only value of n is 88.

Can the number of trailing zeroes in a factorial be ANY possible whole numbers?

While considering successive factorials, $(n-1)!$ and $n!$, the number of trailing zeroes remains same or increases and sometimes it increases by 2 or 3 or more. Thus, the number of trailing zeroes cannot be ALL whole numbers, there would be some misses (when the number of trailing zeroes increases by 2 or 3 or more).

Each of 1! to 4! would have no trailing zeroes

Each of 5! to 9! would have 1 trailing zero

Each of 10! to 14! would have 2 trailing zeros

Each of 15! to 19! would have 3 trailing zeros

Each of 20! to 24! would have 4 trailing zeros

But, each of 25! to 29! would have 6 trailing zeroes.

As we look at higher factorials, the number of trailing zeroes would only increase. Thus there is no factorial which has 5 trailing zeroes i.e. the trailing zeroes in a factorial can never be 5.

E.g. 8: Which of the following cannot be the number of zero at the end of n!?

a. 40   b. 41   c. 42   d. 43

Since the number of trailing zeroes given in the options is in the vicinity of 40, the value of n would be lesser than 40th multiple of 5 i.e. less than 200.

We also know that the number of trailing zeroes in $(n-1)!$ and $n!$ would increment by more than 1 (i.e. miss out one possible whole number) only when n is a multiple of 25.

Thus, checking number of trailing zeroes in 175!, we find that the number of zeroes is $35 + 7 + 1 = 43$.

174! would have 2 lesser trailing zeroes, i.e. 41 trailing zeroes, because we have excluded the number 175, which contributes two 5's.

Thus, the number of trailing zeroes in any factorial would never be 42.

(each of 174!, 173!, ..., 170! would have 41 trailing zeroes and each of 169!, 168!, ..., 165! would have 40 trailing zeroes, the one 5 contributed by 170 being excluded).

Exercise

16. A person starts multiplying consecutive positive integers from 20. How many numbers should he multiply before he will have a result that will end with 3 zeroes?

1. 2 2. 3 3. 4 4. 5 5. 6

17. Find the number of trailing zeroes when all the numbers from 1 to 100 are multiplied together. Also find the number of trailing zeroes if all the numbers from 101 to 200 are multiplied together.

1. 24, 48 2. 24, 49 3. 11, 11 4. 24, 24 5. 24, 25

18. Find the number of zeroes at the end of 250!.

1. 25 2. 27 3. 31 4. 62 5. 75

19. Find the number of zeroes at the end of 125! + 100!. And also the number of zeroes at the end of 125! × 100!.

1. 24, 31 2. 31, 55 3. 24, 55 4. 55, 24 × 31 5. 55, 24^{31}

20. Find the number of zeroes at the end of $(25!)^{5!}$.

1. 720 2. 6^{10} 3. 6^{120} 4. 6^{12} 5. None of these

21. Which of the following could represent the exact number of zeroes that *n*! could end with, for any natural value of *n*?

1. 17 2. 30 3. 29 4. 32 5. 23

22. Which of the following cannot be the number of zeroes at the end of any factorial?

1. 63 2. 61 3. 59 4. 57 5. 56

23. How many of the first 20 natural numbers could NOT be the number of trailing zeroes of *n*!, where *n* is any natural number?

1. None 2. 1 3. 2 4. 3 5. 4

24. Which of the following is the number of zeroes in the product:
$1^1 \times 2^2 \times 3^3 \times 4^4 \times 99^{99} \times 100^{100}$.

1. 100 2. 1100 3. 1200 4. 1300 5. 2300

25. If *n*! and (4*n*)! end with 25 and 106 zeroes respectively, which of the following is a value that *n* can take?

1. 105 2. 106 3. 107 4. 108 5. 109

Factorisation

In classification, we have seen that few natural numbers are classified as primes viz. 2, 3, 5, 7, 11, 13, …… These are numbers that cannot be written as a product of two or more smaller numbers. So we can think of these numbers as 'indestructible'. All other numbers, namely composite numbers (the number 1 is not part of this entire discussion), can be written as product of smaller numbers.

E.g.: $4 = 2 \times 2$; $6 = 2 \times 3$; $8 = 2 \times 4$; $72 = 8 \times 9$.

Few of these smaller multiplicands can themselves be 'broken down' into products of yet smaller numbers …

E.g. $8 = 2 \times 4 = 2 \times (2 \times 2)$; $72 = 8 \times 9 = (2 \times 4) \times (3 \times 3)$.

The process can be continued till the numbers can no longer be 'broken down' …

E.g. $72 = 8 \times 9 = (2 \times 4) \times (3 \times 3) = 2 \times 2 \times 2 \times 3 \times 3$.

Now since each of the multiplicands is a prime number, they can no longer be broken down into product of smaller numbers.

This process of writing a number as a product of prime numbers is called factorization.

Primes as 'Atoms' of Number System

One way of understanding primes is that they are similar to 'Atoms' i.e. they are the 'basic', 'indestructible' 'components' of any number. And that all other numbers are made up of various of these 'atoms' combining in various quantities. Thus, one should view larger numbers as follows …

$36 = 2 \times 2 \times 3 \times 3$

$60 = 2 \times 2 \times 3 \times 5$

Writing 12 as 3×4 is not complete factorization because 4 is a composite number and not prime. The complete factorization is writing 12 as $2 \times 2 \times 3$, where all the multiplicands are prime.

Finally all the like primes are collected together and are written in the exponent form and this is called the factorised form, e.g. the factorised form of 72 is $2^3 \times 3^2$.

The general Factorised Form

The factorised form of a number is written in a general case as $p_1^a \times p_2^b \times p_3^c \times \ldots\ldots$ where p_1, p_2, p_3 are different prime numbers and a, b, c are their respective powers.

Few factorised form are:

$16 = 2^4$ $\quad$ $24 = 2^3 \times 3$

$15 = 3 \times 5$ $\quad$ $72 = 8 \times 9 = 2^3 \times 3^2$

$84 = 12 \times 7 = 2^2 \times 3 \times 7$ $\quad$ $198 = 9 \times 22 = 2 \times 3^2 \times 11$

Next let's learn how to factorise a large number, say 7920

In primary school we learnt the following process ...

$$\begin{array}{r|l} 2 & \underline{7920} \\ 2 & \underline{3960} \\ 2 & \underline{1980} \end{array}$$

...... and so on

We cannot afford to do this process in the exam as it takes far too much time (and we are no longer in primary school)

First Take: On first look we should try to break the number into two large parts (and not as 2×3960). The number 7920 can easily be thought as 10×792.

Second Take: Now, factorise the smaller multiplicands using Divisibility rules and trying larger numbers, typically 8, 9 or 12.

10 is known as 2×5 and so factorizing it is not a problem. While factorizing 792, again resist the temptation to write it as 2×396. It should be pretty obvious that 792 is divisible by 9 (as sum of digits is divisible by 9) and so factorise 792 as 9×88.

Now the number 7920 can be written in factorised form as $2 \times 5 \times 9 \times 8 \times 11$ i.e. $2^4 \times 3^2 \times 5 \times 11$.

Factorising a larger number

If the number is divisible by 2, check if it is divisible by 4 (last two digits divisible by 4) and if it is divisible by 8 (last three digits divisible by 8).

If number is divisible by 3, do check if it is divisible by 9 (sum of digits divisible by 9)

If number ends with 5, 5 is surely a factor; if number ends with 25 or 50 or 75 or 00, then 25 is definitely a factor.

Thus, in the first step, try to break the number into larger factors, rather than breaking it in 2's and 3's.

Factorise 572.

We see that 572 is divisible by 4 but not by 8. Thus 2^2 will be a factor. Reducing the number by 4, we get $572 = 4 \times 143$. Now we know that 143 is 11×13, both of which are prime and cannot be further factorised. Thus, $572 = 2^2 \times 11 \times 13$.

Factorise 2016.

This number is divisible by 8. So reducing, we have 8 × 252.

252 is further divisible by 4. Reducing further, 252 = 4 × 63.

Now we know 63 is 9 × 7. Thus we have the factorised form as $2^5 \times 3^2 \times 7$.

Uses of factorised form

A student who is comfortable with Math usually discounts factorisation as an easy topic. We strongly recommend not to ignore the factorised form. Not only can tough questions be based on the factorised form, but more importantly the form tells us a lot about the number. Whenever any property of a number is given or asked, it can be traced back to the factorised form and this is a very good way to analyse and learn more of numbers.

Prime Factors of the number:

Consider the number $2^4 \times 3^2 \times 5^2 \times 13$.

The only prime factors of this number are 2, 3, 5 and 13. Thus, the number is not divisible by 7 or 11 or 17 and so on and nor by any multiple of them.

E.g. 1: How many three digit numbers are such that each of 3, 5 and 7 are factors of the number and that they are the ONLY prime factors of the number?

Since each of 3, 5 and 7 are factors of the number, the exponents of each of 3, 5 and 7 in the factorised form has to be atleast 1. Further, since no other prime is a factor, no other prime i.e. 2, 11, 13, ... should be present in the factorised form. Thus, the number has to be of the form $3^{\geq 1} \times 5^{\geq 1} \times 7^{\geq 1}$

The smallest such number is 3 × 5 × 7 = 105 and further such numbers will be arrived at when 105 is multiplied further with 3, 5, 7 or powers of them.

Since we need only three digit numbers, the numbers can easily be found as 105 × 3 = 415; 105 × 9 = 945; 105 × 5 = 525; 105 × 7 = 735. All other multiples with higher powers of 3, 5, or 7 will be more than three digit numbers. Thus there are 5 such numbers.

Odd or Even Numbers:

A number is even only when it is divisible by 2 (in other words is a multiple of 2). Thus the factorised form of an even number would definitely have 2 as one of the prime factors. The factorised form of an even number would be $2^{\geq 1} \times p_2^{\,b} \times p_3^{\,c} \times \ldots\ldots$ i.e. the power of 2 has to be 1 or greater (cannot be zero).

Similarly, in the factorised form of an odd number, $p_1^a \times p_2^b \times p_3^c \times ......$, none of the prime p_1, p_2, p_3, ... is going to be 2.

If the number $2^a \times 3^b \times 5^c \times ......$ is odd, then a necessarily has to be zero.

E.g. 2: With what least number should 13440 be divided so that the quotient is a odd number?

Factorising 13440, we see that the number is divisible by 8. Thus the number can be broken as 8 × 1680. Again 1680 is divisible 8. Thus, 13440 = 8 × 8 × 210. Since 210 = 3 × 7 × 10, the factorised form of 13440 is $2^7 \times 3 \times 5 \times 7$.

This has to be divided by a number such that the quotient is odd i.e. the quotient does not have any 2's in its factorised form. Thus, we have to get rid of the 2^7 that is currently present and so the least number with which 13440 has to be divided to get a odd quotient is 2^7 i.e. 128.

Perfect Squares, Perfect Cubes, Perfect higher powers:

Perfect squares are squares of natural numbers. Thus consecutive squares are 1, 4, 9, 16, 25, 36, 49, ... Alternately, perfect squares are those numbers whose square root is a natural number.

For the number, $\left(p_1^a \times p_2^b \times p_3^c \times ......\right)^{\frac{1}{2}}$, to be a natural number, each of a, b, c should be divisible by 2 i.e. each of a, b, c should be even.

Similarly $p_1^a \times p_2^b \times p_3^c \times ......$ will be a perfect cube if and only if each of a, b, c is divisible by 3 i.e. is a multiple of 3.

Similar conditions can be framed for perfect higher powers.

E.g. 3: With what least number should 864 be multiplied so that it is a perfect square as well as a perfect cube?

A number $p_1^a \times p_2^b \times p_3^c \times ......$ would be a perfect square as well as a perfect cube only when each of a, b, c, is a multiple of 2 as well as of 3 i.e. a multiple of 6.

To gain more insights into 864, we should factorise it. It is obvious that the number is divisible by 8. Thus 864 = 8 × 108. And 108 is divisible by 9. Thus 864 = 8 × 9 × 12 and from here we can write the factorised form directly as $2^5 \times 3^3$.

Now $2^5 \times 3^3$ has to be multiplied with a number (i.e. exponents have to be increased) such that the exponent of 2 and 3 becomes a multiple of 6. Thus the required number is 2×3^3 i.e. 54.

E.g. 4: With what least number should 40500 be divided so that the quotient is odd and is a perfect cube?

$40500 = 405 \times 100 = 9 \times 45 \times 25 \times 4$.

Thus the factorised form of 40500 is $2^2 \times 3^4 \times 5^3$. This number has to be divided by another number (i.e. the exponents have to be reduced) so that the exponent of 2 is zero (for quotient to be odd) and exponent of 3 and 5 has to be a multiple of 3 (for quotient to be a perfect cube). Thus we have to divide by $2^2 \times 3$ i.e. 12.

Perfect Squares/Cubes dividing/being divided:

The only numbers that divide 2^4 are 2^0 or 2^1 or 2^2 or 2^3 or 2^4. The exponent of 2 in the divisor cannot be more than 4 or else it would not divide 2^4.

Does the denominator cancel out completely?

When we say that a divides b, we mean that a divides b 'completely' i.e. $\frac{b}{a}$ is a natural number and is not a decimal.

Speaking loosely, this means 'In the division, $\frac{b}{a}$, the denominator cancels out completely with the numerator'.

If a term is left in the denominator and it cannot be cancelled with any term of the numerator, then a does not divide b; a is not a factor of b; b is not a multiple of a.

Thus, 3 divides 39 because $\frac{\cancel{39}}{\cancel{3}} = 13$, the denominator cancels out totally.

But consider the division $\frac{2^4 \times 3^2}{2^3 \times 5}$. On cancelling out terms we have $\frac{2^{\cancel{4}\,1} \times 3^2}{2^{\cancel{3}} \times 5} = \frac{2 \times 3^2}{5}$. The 5 in the denominator cannot be cancelled out with any term of the numerator. Thus, the numerator is not completely divisible by the denominator.

Similarly, when we try to divide 2^4 with 2^5, we get $\frac{2^{\cancel{4}}}{2^{\cancel{5}\,1}} = \frac{1}{2}$ and since the 2 in the denominator cannot be cancelled with the numerator, we say that 2^5 cannot divide 2^4.

Say, we want to find the highest perfect square that can divide $2^5 \times 3^2 \times 5^3$.

The number that would divide $2^5 \times 3^2 \times 5^3$ has to necessarily have only 2, 3 and 5 as its prime factors i.e. the number cannot be a multiple of 7 or 11 or 13 or higher primes (none of these primes would cancel with $2^5 \times 3^2 \times 5^3$).

Thus, a number dividing $2^5 \times 3^2 \times 5^3$ is of the form $2^a \times 3^b \times 5^c$ such that a has to be less than or equal to 5, b has to be less than or equal to 2 and c has to be less than or equal to 3.

Further we want the factor to be the 'highest' possible 'square' i.e. the exponents *a*, *b*, *c* have to be as high as possible and should also be even, so that the number is a square. With the given conditions, the possible value are $a = 4$, $b = 2$ and $c = 2$. Thus the highest square that can divide $2^5 \times 3^2 \times 5^3$ is $2^4 \times 3^2 \times 5^2$ i.e. $16 \times 9 \times 25 = 3600$.

E.g. 5: How many perfect cubes can divide 51840?

By visual check we can see that 51840 is divisible by 10 and also 9. Thus, $51840 = 9 \times 576 \times 10$. And we know that 576 is square of 24. Thus, the factorised form of 51840 is $2^7 \times 3^4 \times 5$.

We are searching for a number such that in the division $\frac{2^7 \times 3^4 \times 5}{\text{the number}}$, the denominator gets completely cancelled. The number has to be of the form $2^{\le 7} \times 3^{\le 4} \times 5^{\le 1}$. Also the number has to be a cube i.e. the exponents have to be a multiple of 3. Thus, the exponent of 2 could take 3 different values viz. 0, 3, 6; the exponent of 3 could take 2 different values viz. 0 and 3; the exponent of 5 could only be 0. Combining various possibilities we find there are $3 \times 2 \times 1 = 6$ possibilities, viz.

$2^6 \times 3^3$; $2^3 \times 3^3$; $2^0 \times 3^3$;

$2^6 \times 3^0$; $2^3 \times 3^0$; $2^0 \times 3^0$.

These are the 6 cubes that divide 51840.

E.g. 6: Find the least perfect square that can be completely divided by 51840?

This example is different from the earlier one – in this question we are looking for a number that is divisible by 51840, whereas in the earlier example, 51840 was being divided.

Thus, we are looking for a number such that in the division $\frac{\text{the number}}{2^7 \times 3^4 \times 5}$, the denominator gets completely cancelled. Further the exponents of the primes in the numerator are all even, because the number has to be a square. The least possible such number is $2^8 \times 3^4 \times 5^2$.

Number of trailing zeroes in a number:

A trailing zero would be present in a number only when there is a 5 AND a 2 present in the number. In fact, EACH pair of a 5 and a 2 present in the number would result in one trailing zero.

Thus, the number $2^1 \times 5^1 \times p_3^{\ c} \times p_4^{\ d} \times ......$ (where p_3, p_4 are primes other than 2 and 5) would have exactly one trailing zero whereas the number $2^2 \times 5^2 \times p_3^{\ c} \times p_4^{\ d} \times ......$ would have exactly two trailing zeroes. Similarly the number $2^2 \times 5^1 \times p_3^{\ c} \times p_4^{\ d} \times ......$ would have only 1 trailing zeroes. Even though there are two 2's present, there is only one 5 and thus only one pair of 2 and 5 can be formed.

E.g. 7: Find the number of trailing zeroes in the product $375^5 \times 12^7$

Factorising, $375^5 \times 12^7 = (125 \times 3)^5 \times (4 \times 3)^7 = 5^{15} \times 3^5 \times 2^{14} \times 3^7 = 2^{14} \times 5^{15} \times 3^{12}$

The number of pairs, (2 × 5), that can be formed is 14 and this will be the number of trailing zeroes in the product.

E.g. 8: If the number $\frac{8^{16-a}}{25^{3-a}}$ is divisible by 10^{12}, find the number of distinct values that a can assume.

$$\frac{8^{16-a}}{25^{3-a}} = 2^{48-3a} \times 5^{2a-6}$$

For this number of be divisible by 10^{12} i.e. by $2^{12} \times 5^{12}$, the exponent of each of 2 and 5 should be 12 or higher. Thus,

$48 - 3a \geq 12$ and $2a - 6 \geq 12$

$\Rightarrow 36 \geq 3a$ and $2a \geq 18$

$\Rightarrow a \leq 12$ and $a \geq 9$.

The values that satisfies both of these conditions are a = 9 or 10 or 11 or 12. Thus, a can assume 4 different values.

Exercise

1. Find the number of prime factors of 17!.

   1. 17   2. 13   3. 11   4. 8   5. 7

2. With what least number should 720 be multiplied to result in a perfect cube?

   1. 12   2. 36   3. 225   4. 300   5. 360

3. What is the highest perfect square that can divide 384?

   1. 16   2. 36   3. 64   4. 81   5. None of these

4. Find the sum of the digits of the least number n, such that $2n$ is a square and $3n$ is a cube.

   1. 9   2. 10   3. 11   4. 12   5. 13

5. With what least number should 13440 be divided so that the quotient is not a multiple of 20?

   1. 5   2. 10   3. 40   4. 240   5. 320

6. Find the number of zeroes at the end of $2^7 \times 3^5 \times 5^6 \times 7^3 \times 14^6 \times 25^3$

   1. 5   2. 6   3. 7   4. 12   5. 13

7. With what least number should $11^{13} \times 9^{10}$ be divided to result in a perfect cube?

   1. 9   2. 11   3. 81   4. 99   5. 121

8. For how many values of a, is the number $\dfrac{15^{a+3} \times 28^{8-a}}{35^{2a-9}}$ is an integer.

   1. 7   2. 8   3. 9   4. 10   5. 11

Number of factors

Let's say we want to find the number of factors of $2^3 \times 3^2$.

Factors are those that divide the given number completely. It is obvious that no multiple of 5, 7, 11, or other higher primes can divide the given number.

Further, the number is divisible by each of 2^0, 2^1,2^2, 2^3 but not by any higher power of 2 like 2^4, 2^5, …… Also the number is divisible by each of 3^0, 3^1, 3^2 but not by 3^3, 3^4, …….

We should also realise that the number can be divisible by any 'combination' of one of {2^0, 2^1,2^2, 2^3} and one of {3^0, 3^1, 3^2} i.e. by numbers of the type $2^1 \times 3^2$ or $2^2 \times 3^1$ or $2^1 \times 3^1$ and so on.

Thus, with 2^0, we could have a total of 3 'combinations' i.e.

$$2^0 \begin{cases} 3^0 & = 2^0 \times 3^0 \\ 3^1 & = 2^0 \times 3^1 \\ 3^2 & = 2^0 \times 3^2 \end{cases}$$

Each of these 3 numbers would divide $2^3 \times 3^2$ and thus would be a factor.

Similarly with 2^1, we could have 3 more combinations ……

$$2^1 \begin{cases} 3^0 & = 2^1 \times 3^0 \\ 3^1 & = 2^1 \times 3^1 \\ 3^2 & = 2^1 \times 3^2 \end{cases}$$

…… and each of these 3 factors would be distinct from the earlier 3 factors.

Similarly with EACH of 2^2 and 2^3, we would get 3 more distinct factors and thus the total number of factors would be $4 \times 3 = 12$.

Consider another example: find the number of factors of $7^3 \times 13 \times 29^2$.

Factors of the given number can only be 'combinations' of selected powers of 7, 13 and 29. The exponent of 7 could be any of 0, 1, 2 or 3; the exponent of 13 could be only 0 or 1; and the exponent of 29 could be 0, 1 or 2. Any other higher power of 7 or 13 or 29 would not be able to completely divide the given number.

Thus the total possible 'combinations' could be as follows ….

- 7^0
  - 13^0
    - $29^0 = 7^0 \times 13^0 \times 29^0$
    - $29^1 = 7^0 \times 13^0 \times 29^1$
    - $29^2 = 7^0 \times 13^0 \times 29^2$
  - 13^1
    - $29^0 = 7^0 \times 13^1 \times 29^0$
    - $29^1 = 7^0 \times 13^1 \times 29^1$
    - $29^2 = 7^0 \times 13^1 \times 29^2$
- 7^1
  - 13^0
    - $29^0 = 7^1 \times 13^0 \times 29^0$
    - $29^1 = 7^1 \times 13^0 \times 29^1$
    - $29^2 = 7^1 \times 13^0 \times 29^2$
  - 13^1
    - $29^0 = 7^1 \times 13^1 \times 29^0$
    - $29^1 = 7^1 \times 13^1 \times 29^1$
    - $29^2 = 7^1 \times 13^1 \times 29^2$
- 7^2
  - 13^0
    - $29^0 = 7^2 \times 13^0 \times 29^0$
    - $29^1 = 7^2 \times 13^0 \times 29^1$
    - $29^2 = 7^2 \times 13^0 \times 29^2$
  - 13^1
    - $29^0 = 7^2 \times 13^1 \times 29^0$
    - $29^1 = 7^2 \times 13^1 \times 29^1$
    - $29^2 = 7^2 \times 13^1 \times 29^2$
- 7^3
  - 13^0
    - $29^0 = 7^3 \times 13^0 \times 29^0$
    - $29^1 = 7^3 \times 13^0 \times 29^1$
    - $29^2 = 7^3 \times 13^0 \times 29^2$
  - 13^1
    - $29^0 = 7^3 \times 13^1 \times 29^0$
    - $29^1 = 7^3 \times 13^1 \times 29^1$
    - $29^2 = 7^3 \times 13^1 \times 29^2$

If observed carefully, each of the factors is a distinct number because the power of 7 or 13 or 29 differ in each of the combinations.

Since the exponent of 7 could assume 4 different values (from 0 to 3), the exponent of 13 could assume 2 distinct values (from 0 to 1) and the exponent of 29 could assume 3 distinct values (from 0 to 2), the total number of combinations are $4 \times 2 \times 3 = 24$.

Having seen the above two examples, one should easily understand that the factors of $2^5 \times 3^2 \times 7^3$ would be only of the form $2^a \times 3^b \times 7^c$ such that a could assume any value from 0 to 5 i.e. 6 different values, b could assume any value from 0 to 2 i.e. 3 different values and c could assume any value from 0 to 3 i.e. 4 different values. Thus the various combinations and the number of factors would be $6 \times 3 \times 4$ i.e. 72.

Procedure to find the number of factors

Step 1: Factorise the given number, say $p_1^a \times p_2^b \times p_3^c \times ......$

Step 2: The number of factors is $(a+1)\times(b+1)\times(c+1)\times ......$

The reason we have $(a + 1)$, $(b + 1)$, $(c + 1)$, ... in the multiplicand is because any combination of the type $p_1^{0 \text{ to } a} \times p_2^{0 \text{ to } b} \times p_3^{0 \text{ to } c} \times ......$ would be a factor and the number of distinct values from 0 to a is $(a + 1)$, the number of distinct values from 0 to b is $(b + 1)$ and so on.

Questions might ask us to find the number of factors that are even (or odd), or that are perfect squares (or cubes) or any such condition. A good way to approach such questions is to assume factor of $p_1^a \times p_2^b \times p_3^c$ .... as $p_1^{0 \text{ to } a} \times p_2^{0 \text{ to } b} \times p_3^{0 \text{ to } c}$ .... Further the various conditions for a number to be even, odd, perfect square, etc that were learnt have to be applied as shown in the following examples......

E.g. 1: Find the number of factors of $2^7 \times 3^4 \times 7^3$ that are even.

A factor of $2^7 \times 3^4 \times 7^3$ is of the type $2^{0 \text{ to } 7} \times 3^{0 \text{ to } 4} \times 7^{0 \text{ to } 3}$.

For the factor to be even, the exponent of 2 has to be atleast 1. Thus, an additional condition is imposed because the factor has to be even. Thus, the required factors could now only be of the form $2^{1 \text{ to } 7} \times 3^{0 \text{ to } 4} \times 7^{0 \text{ to } 3}$.

Since the exponent of 2 can assume 7 distinct values, exponent of 3 can assume 5 distinct values and the exponent of 7 can assume 4 distinct values, the number of factors that are even are 7 × 5 × 4 = 140.

E.g. 2: Find the number of factors of $2^7 \times 3^4 \times 7^3$ that are perfect squares.

A factor of $2^7 \times 3^4 \times 7^3$ is of the type $2^{0 \text{ to } 7} \times 3^{0 \text{ to } 4} \times 7^{0 \text{ to } 3}$.

For the factor to be perfect square as well, the exponent of each of 2, 3, and 7 has to be an even number. Thus, an additional condition is imposed because the factor has to be perfect square. Thus, the required factors could now only be of the form $2^{0 \text{ or } 2 \text{ or } 4 \text{ or } 6} \times 3^{0 \text{ or } 2 \text{ or } 4} \times 7^{o \text{ or } 2}$.

Since the exponent of 2 can assume 4 distinct values, exponent of 3 can assume 3 distinct values and the exponent of 7 can assume 2 distinct values, the number of factors that are perfect squares are 4 × 3 × 2 = 24.

E.g. 3: Find the number of factors of $2^7 \times 3^4 \times 7^3$ that are multiples of 24.

A factor of $2^7 \times 3^4 \times 7^3$ is of the type $2^{0 \text{ to } 7} \times 3^{0 \text{ to } 4} \times 7^{0 \text{ to } 3}$.

For the factor to be a multiple of 24 i.e. $2^3 \times 3$, the exponent of 2 has to be atleast 3 and that of 3 has to be atleast 1. Thus, the required factors could now only be of the form $2^{3 \text{ to } 7} \times 3^{1 \text{ to } 4} \times 7^{0 \text{ to } 3}$.

Since the exponent of 2 can assume 5 distinct values (viz. 3, 4, 5, 6, 7), exponent of 3 can assume 4 distinct values and the exponent of 7 can assume 4 distinct values, the number of factors that are multiples of 24 are 5 × 4 × 4 = 80.

E.g. 4: Find the number of factors of 6912 that are also the factors of 3888.

Factorising the given numbers ......

Since 6912 is divisible by 8,

$6912 = 8 \times 864 = 8 \times 8 \times 108 = 8 \times 8 \times 4 \times 27 = 2^8 \times 3^3$.

Similalry, $3888 = 8 \times 486 = 8 \times 2 \times 243 = 2^4 \times 3^5$.

A factor of both $2^8 \times 3^3$ and $2^4 \times 3^5$ necessarily has to be of the form $2^{0 \text{ to } 4} \times 3^{0 \text{ to } 3}$, else it would not divide both the numbers. Thus the number of common factors is $5 \times 4 = 20$.

Exercise

1. Find the number of factors of $2^3 \times 3^2 \times 6^4$

   1. 24   2. 30   3. 42   4. 56   5. 60

Directions for questions 2 to 10: Consider the number, $2^4 \times 3^7 \times 5^2$. Of the 120 factors that this number has,

2. How many are odd?

   1. 24   2. 30   3. 42   4. 56   5. 60

3. How many are even?

   1. 60   2. 75   3. 96   4. 105   5. 112

4. How many are perfect squares?

   1. 6   2. 12   3. 18   4. 24   5. 30

5. How many of them have the units digit equal to zero?

   1. 24   2. 32   3. 40   4. 56   5. 64

6. How many of them are composite?

   1. 117   2. 116   3. 115   4. 114   5. 113

7. How many of them are perfect cubes?

   1. None   2. 2   3. 3   4. 6   5. 12

8. How many of them are multiples of $2 \times 3 \times 5$?

   1. 24   2. 32   3. 40   4. 56   5. 64

9. How many of them are perfect squares as well as perfect cubes?

   1. None   2. 1   3. 2   4. 3   5. 4

10. How many of them have their unit digit equal to 5?

   1. 8   2. 10   3. 12   4. 14   5. 16

11. Find the number of common factors of 1080, 1440 and 1800.

   1. 6   2. 12   3. 18   4. 24   5. 30

12. Find the number of factors of $3^5 \times 5^3$ that are not common to the factors of $3^2 \times 5$.

   1. 13   2. 18   3. 6   4. 12   5. None of these

13. How many factors of $12^4 \times 15^3$ and are also multiples of $6^2 \times 4$?

   1. 28   2. 32   3. 35   4. 120   5. 175

14. The number $2^a \times 3^b$, where a and b are natural numbers, has a total of 12 factors. How many distinct values can the number assume?

   1. 2   2. 3   3. 4   4. 5   5. 6

15. Find the ratio of the number of factors of 16! to that of 15!.

   1. 16 : 15   2. 4 : 1   3. 3 : 1   4. 4 : 3   5. 15 : 11

Puzzle

Along a corridor are 1000 doors marked sequentially as 1, 2, 3, 4, ..., 1000. All the doors are currently closed.

Person #1 passes along the corridor and opens all doors. (Now all doors are open)

Next, person #2 passes along the corridor and closes door numbers 2, 4, 6, 8, ... (Now doors are alternately open and close)

Next person #3 passes along the corridor and 'changes the state' of door numbers 3, 6, 9, 12, ... ('changes the state' means if a door is open then he closes it and if a door is closed then he opens it)

Next person #4 passes along the corridor and changes the state of door numbers 4, 8, 12, 16, ...

In a similar manner persons keep changing the state of door numbers which are multiples of the person number, untill the person #1000 changes the state of door number 1000.

How many doors are now open and which doors are they?

Identifying a number given its number of factors

So far we were given a number and then had to find the number of factors. If the number of factors is given, can we find the factorised form of the number?

We have already learnt that the factorised form of any number is of the form

$p_1^a \times p_2^b \times p_3^c \times ......$, where p_1, p_2, p_3, ... are all distinct prime numbers. We also know that the number of factors of such a number is $(a+1)\times(b+1)\times(c+1)\times......$

Consider a number has 6 factors. Then, $(a+1)\times(b+1)\times(c+1)\times...... = 6$. This could have multiple solutions for a, b, c, ... But 6 has to be written as a product of natural numbers and there are limited ways it can be done, namely 6 × 1 × 1 × ...... or 2 × 3 × 1 × 1 .... And no other way. (It does not matter which of a, b, c, ... is 6 and which are 1 i.e. all of 6 × 1 × 1 × ......; 1 × 6 × 1 × ......; or 1 × 1 × 6 × ...... would lead to the same factorised form)

If $(a+1)\times(b+1)\times(c+1)\times...... = 6\times1\times1\times......$, one of a, b, c, ... is 5 and rest all are zeroes and the number is of the form p_1^5, where p_1 could be any prime number.

If $(a+1)\times(b+1)\times(c+1)\times...... = 2\times3\times1\times......$, of a, b, c, ... any one of them is 1, another one is 2 and rest are all zeroes. Thus number is of the form, $p_1^1 \times p_2^2$, where p_1 and p_2 are any two distinct prime numbers.

Thus a number having 6 factors necessarily has to be of the form p_1^5 or $p_1^1 \times p_2^2$.

E.g. 5: How many numbers less than 10,000 have exactly 5 factors?

For the number $p_1^a \times p_2^b \times p_3^c \times......$ to have 5 factors,

$(a+1)\times(b+1)\times(c+1)\times...... = 5$ and this is possible only if one of a, b, c, ... is 4 and rest all are zeroes. Thus the number has to necessarily be of the form

p^4, where p could be any prime number.

Thus, numbers having 5 factors are 2^4, 3^4, 5^4, 7^4, 11^4, 13^4, 17^4, ...

We also have a condition that the number should be less than 10,000. Now, $7^4 = 2401$ and $11^4 = 121^2 = 14{,}641$. Thus all of 11^4, 13^4, 17^4, ... will be greater than 10,000 and the required answer is 4 viz. 2^4, 3^4, 5^4 and 7^4.

E.g. 6: If n is a number having 11 factors, find the number of factors of n^2.

Since 11 is prime and cannot be written as product of natural numbers other than 1, hence a number having 11 factors necessarily has to be of the form p^{10}. And square of this number will be p^{20}. Thus, n^2 will have 21 factors.

E.g. 7: If n^3 has 28 factors, find the number of factors of n.

28 can be written as a product of natural numbers in quite a few ways. Taking each case ......

Case i: $28 = 28 \times 1 \times 1 \times ...... \Rightarrow n^3 = p^{27} \Rightarrow n = p^9$

Writing 28 as a product of 2 natural numbers...

Case ii: $28 = 2 \times 14 \Rightarrow n^3 = p_1 \times p_2^{13}$, which is not possible because all powers of primes numbers in a cube have to be multiples of 3.

Case iii: $28 = 4 \times 7 \Rightarrow n^3 = p_1^3 \times p_2^6 \Rightarrow n = p_1 \times p_2^2$

There is no other way of writing 28 as a product of two natural numbers. Writing 28 as a product of three natural numbers ...

Case iv: $28 = 2 \times 2 \times 7 \Rightarrow n^3 = p_1^1 \times p_2^1 \times p_3^6$, which is again not possible as powers of all the prime numbers are not multiples of 3.

28 cannot be written as a product in any other way.

Thus, if n^3 has 28 factors, then n necessarily has to be p^9 or $p_1 \times p_2^2$. Thus n could have 10 or 6 factors.

E.g. 8: If n has 24 factors, $2 \times n$ has 30 factors and $3 \times n$ has 32 factors, find the number of factors of $12 \times n$.

Since here we have n, $2 \times n$, $3 \times n$, let's assume n as $2^a \times 3^b \times p_3^c \times p_4^d ......$

$$n = 2^a \times 3^b \times p_3^c \times ...... \qquad \Rightarrow (a+1)\times(b+1)\times(c+1)\times(d+1)\times ... = 24 \quad ... (1)$$

$$2 \times n = 2^{a+1} \times 3^b \times p_3^c \times ...... \qquad \Rightarrow (a+2)\times(b+1)\times(c+1)\times(d+1)\times ... = 30 \quad ... (2)$$

$$3 \times n = 2^a \times 3^{b+1} \times p_3^c \times ...... \qquad \Rightarrow (a+1)\times(b+2)\times(c+1)\times(d+1)\times ... = 32 \quad ...(3)$$

$$(2) \div (1) \qquad \Rightarrow \frac{(a+2)}{(a+1)} = \frac{5}{4} \Rightarrow a = 3$$

$$(3) \div (1) \qquad \Rightarrow \frac{(b+2)}{(b+1)} = \frac{4}{3} \Rightarrow b = 2$$

Substituting values of a and b in any of (1), (2) or (3), we get,
$(c+1)\times(d+1)\times ... = 2$

$$12 \times n = 2^{a+2} \times 3^{b+1} \times p_3^c \times ...... \Rightarrow (a+3)\times(b+2)\times(c+1)\times(d+1)\times ... = 6 \times 4 \times 2 = 48$$

Thus, $12 \times n$ would have 48 factors.

Short-cut:

While finding number of factors of n and $2 \times n$, the only change in $(a+1)\times(b+1)\times(c+1)\times......$ is that instead of $(a + 1)$, now we would have $(a + 2)$. This causes the number of factors to change from 24 to 30. Thus, $\frac{(a+2)}{(a+1)} = \frac{5}{4} \Rightarrow a = 3$. In $12 \times n$, we increase the power of 2 by 2. Thus, instead of $(a + 1)$, we would have $(a + 3)$. Thus number of factors would get multiplied with $\frac{(a+3)}{(a+1)} = \frac{6}{4}$ *i.e.* $\frac{3}{2}$

While finding number of factors of n and $3 \times n$, the only change in $(a+1)\times(b+1)\times(c+1)\times......$ is that instead of $(b + 1)$, now we would have $(b + 2)$. This causes the number of factors to change from 24 to 32. Thus, $\frac{(b+2)}{(b+1)} = \frac{4}{3}$. In $12 \times n$ also we would increase the power of 3 by 1. Thus, number of factors would get multiplied with $\frac{(b+2)}{(b+1)} = \frac{4}{3}$.

Thus, $12 \times n$ would have $24 \times \frac{3}{2} \times \frac{4}{3} = 48$ factors.

Exercise

16. How many two digit numbers have exactly 3 factors?

    1. 9 2. 8 3. 6 4. 4 5. 2

17. Consider $n = 2^a \times 3^b$. If the ratio of the number of factors of n, $2n$ and $3n$ is 12 : 16 : 15, find the number of factors of $6n$.

    1. 6 2. 12 3. 20 4. 24 5. Cannot be determined

18. If the number of factors of a and b are 6 and 8 respectively, find the number of factors of $a \times b$, given that a and b are co-prime numbers.

    1. 48 2. 42 3. 40 4. 35 5. No unique value

19. If the number of factors of n are 15, find the number of factors of $4n$, given that n is an odd number.

    1. 30 2. 45 3. 60 4. 30 or 45 5. 45 or 60

20. If the number of factors of n and $2n$ are 15 and 20 respectively, find the number of factors of $8n$.

    1. 24 2. 25 3. 30 4. 32 5. 36

21. If the number of factors of n^3 is 34, find the number of factors of n^2.

    1. 11 2. 12 3. 22 4. 23 5. Cannot be determined

22. How many two digit numbers have 12 factors?

    1. 1 2. 2 3. 3 4. 4 5. 5

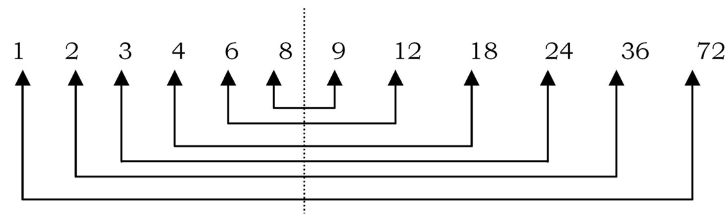
23. If the number of factors of n is 12, how many distinct values can the number of factors of n^2 assume?

    1. 1 2. 2 3. 3 4. 4 5. 5

Relative Placement of Factors

Consider the number $72 = 2^3 \times 3^2$. We know that the number has $4 \times 3 = 12$ factors. These twelve factors when written in ascending order are 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.

An observation worth noting is that product of factors equidistant from the center is 72, the number itself!



And it would obviously be so...... if a is a factor of n, then $\frac{n}{a} = b$, a natural number.

Thus, $\frac{n}{b} = a$ and hence b would also be a factor of n. Further we would have

$a \times b = n$.

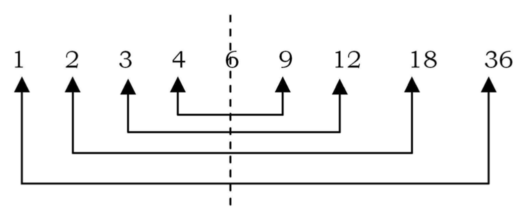
Thus factors will exist in pairs (a, b) such that $a \times b = n$.

Does this mean that the number of factors would always be even? Read on ...

Next consider the factors of 36 i.e. $2^2 \times 3^2$.

The factors in ascending order are 1, 2, 3, 4, 6, 9, 12, 18, 36

Now looking for pairs of factors, a and b such that $a \times b = n$, we have



We also see that this time, there is a natural number which is exactly at the center in the list of factors i.e. the number of factors are odd. Also this number in the center is such that $6 \times 6 = 36$ i.e. the factor multiplied with itself is the given number. But naturally, such a factor is the square root of the given number.

When the number of factors is even or odd ......

The above two examples would make it clear that ...

...... if the number of factors is even, there is no 'center' in the list of factors and thus there is no factor multiplied with itself that results in the given number. Thus if a number has even number of factors, the number is not a perfect square.

...... if the number of factors is odd, one factor occupies the center in the list of factors and this factor multiplied with itself results in the given number. Thus if a number has odd number of factors, the number is a perfect square.

The converse also holds true i.e. if number is a perfect square it has to have odd number of factors and if it is not a perfect square, it has to have even number of factors.

The above can also be proved mathematically ......

The number of factors of $n = p_1^a \times p_2^b \times p_3^c \times ......$ is $(a+1)\times(b+1)\times(c+1)\times ......$

n is a perfect square $\Rightarrow$ each of a, b, c, ... is even

$\Rightarrow$ each of $(a + 1)$, $(b + 1)$, $(c + 1)$, ... is odd

$\Rightarrow$ the product $(a+1)\times(b+1)\times(c+1)\times ...$ is odd

$\Rightarrow$ the number of factors is odd

Conversely,

number of factors being odd $\Rightarrow$ the product $(a+1)\times(b+1)\times(c+1)\times ...$ is odd

$\Rightarrow$ each of $(a + 1)$, $(b + 1)$, $(c + 1)$, ... is odd

$\Rightarrow$ each of a, b, c, ... is even

$\Rightarrow$ the number n is a perfect square.

Few learning from the above placement of factors ......

As seen in the above two figure, when all the factors of n are written in ascending order, $\sqrt{n}$ would be dividing the list into two equal parts.

Further, $\sqrt{n}$ itself would be present in the list at the center if and only if $\sqrt{n}$ is a natural number.

If a and b are two distinct factors of n such that $a \times b = n$, one of a and b will be less than $\sqrt{n}$ and other will be greater than $\sqrt{n}$.

Finding if a number is prime

The fact that factors of n exists in pairs, a and b, such that $a \times b = n$ and one of a and b is less than $\sqrt{n}$ is used to identify if a given number is prime

To find if a given number is prime, we just check if the number is divisible by any prime upto the natural number just less than $\sqrt{n}$. Thus to check if 203 is prime, we just check if 203 is divisible by prime number upto 14 $\left(\because \sqrt{203} \approx 14\right)$.

The question is why do we check only till $\sqrt{n}$? Can there not be a factor of 203 more than 14? If yes, why do we not check for factors higher than 14?

E.g. 9: A perfect square is such that it has exactly 11 factors between 1 and its square root (excluding these). Find the total number of factors of the perfect square.

Since the number is a perfect square, its square root will also be a factor of the number. Seeing the following, it should be obvious that the given perfect square should have 25 factors.

$$1, \underbrace{\qquad\qquad}_{\text{11 factors}}, \sqrt{n}, \underbrace{\qquad\qquad}_{\text{11 factors}}, n$$

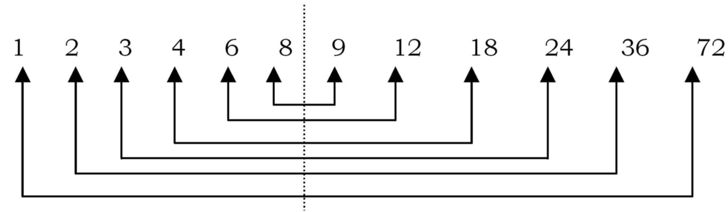
(If the number was not a perfect square and rest all data was same, the number would have 24 factors.)

E.g. 10: If all the factors of 8000 are written in increasing order, from left to right, find the factor which occupies the 25^{th} position from the left end.

$8000 = 8 \times 1000 = 8 \times 8 \times 125 = 2^6 \times 5^3$. Thus it would have $7 \times 4 = 28$ factors. When written increasing order, product of 4^{th} factor and 25^{th} factor would be 8000. Since the first 4 factors are 1, 2, 4, 5, the 25^{th} factor will be $\frac{8000}{5} = 1600$.

Writing a number as product of two integers

Consider the number 72 and its factors as seen above ......



The only ways to write 72 as a factor of two natural numbers is 1 × 72; 2 × 36; 3 × 24; ......; 8 × 9. Thus the number of ways of writing 72 as a product of two natural number is half the number of factors i.e. 12/2 = 6 ways. There could be slight variations to these types of problems as shown in the following examples.

E.g. 11: Find the number of solutions to $a \times b = 324$, where a and b are two distinct natural numbers.

$324 = 18^2 = 2^2 \times 3^4$. Thus, it would have 15 factors. Since it is a perfect square (and number of factors is odd), the middle factor would be 18. And the remaining 14 factors would form 7 pairs of numbers whose product will be 324.

But since in this question, since a and b are mentioned specifically, each pair would result in two solutions for a and b i.e. 2 × 162 could result in $a = 2$, $b = 162$ or $a = 162$, $b = 2$. Thus total number of required solutions is 14 (and not 7).

(If a and b could be identical, there would be 15 possible solutions.)

E.g. 12: In how many ways can 10! be written as a product of two integers.

Factorising 10!, we get $10! = 2^8 \times 3^4 \times 5^2 \times 7$. Thus it would have 9 × 5 × 3 × 2 = 270 factors, which would form 135 pairs, all of whose product would be 10!.

However in this question we need to write 10! as product of INTEGERS, which could also be negatives. Thus each pair, say $\{2^8 \times 3^4, 5^2 \times 7\}$ would result as two pairs, once when both are positive and once when both are negative. Thus, required answer is 270 ways.

The above has an application in numerous tricky questions. While most of these questions start with an algebraic expression, they can immediately be converted to an application of writing a number as a product of two integers.

Writing a number as difference of squares

If the given number is odd

E.g. 13: In how many ways can 675 be written as a difference of two perfect squares.

We need x and y such that $x^2 - y^2 = 675$ i.e. $(x+y)\times(x-y)=675$. Thus, we need to write 675 as a product of two distinct numbers, the larger of which will be $x + y$ and the smaller will be $x - y$.

Factorising, $675 = 25 \times 27 = 3^3 \times 5^2$. Thus 675 will have 12 factors and can be written as a product of two natural numbers in 6 ways. Each of these 6 ways will result in 6 different ways of writing 675 as a difference of two perfect squares.

While the answer to the question is already found, let's go a little further and find the 6 different ways ...

Finding x and y when $x + y$ and $x - y$ is given

If $x + y = 22$ and $x - y = 8$, to find x, just add the RHS and divide by 2 and to find y, subtract the RHS and divide by 2. Thus, $x = \frac{22+8}{2} = 15$ and $y = \frac{22-8}{2} = 7$.

$675 = 1\times675 \quad \Rightarrow x+y=675,\ x-y=1 \quad \Rightarrow x=338,\ y=337$

(Check: $338^2 - 337^2 = (338 + 337) \times (338 - 337) = 675$)

$675 = 3\times225 \quad \Rightarrow x+y=225,\ x-y=3 \quad \Rightarrow x=114,\ y=111$

(Check: $114^2 - 111^2 = 12996 - 12321 = 675$)

$675 = 5\times135 \quad \Rightarrow x+y=135,\ x-y=5 \quad \Rightarrow x=70,\ y=65$

(Check: $70^2 - 65^2 = 4900 - 4225 = 675$)

$675 = 9\times75 \quad \Rightarrow x+y=75,\ x-y=9 \quad \Rightarrow x=42,\ y=33$

(Check: $42^2 - 33^2 = 1764 - 1089 = 675$)

$675 = 15\times45 \quad \Rightarrow x+y=45,\ x-y=15 \quad \Rightarrow x=30,\ y=15$

(Check: $30^2 - 15^2 = 900 - 225 = 675$)

$675 = 25\times27 \quad \Rightarrow x+y=27,\ x-y=25 \quad \Rightarrow x=26,\ y=1$

(Check: $26^2 - 1^2 = 676 - 1 = 675$)

If the given number is even

E.g. 14: In how many ways can 180 be written as a difference of two squares of natural numbers.

We need x and y such that $x^2 - y^2 = 180$ i.e. $(x+y)\times(x-y)=180$. Thus, we need to write 180 as a product of two distinct numbers, the larger of which will be $x + y$ and the smaller will be $x - y$.

Factorising, $180 = 2^2 \times 3^2 \times 5$. Thus 180 will have 18 factors and can be written as a product of two natural numbers in 9 ways.

Would each of these 9 ways will result in 9 different ways of writing 180 as a difference of two perfect squares? Let's see ...

$$180 = 1\times180 \quad \Rightarrow x+y=180,\ x-y=1 \quad \Rightarrow x=\frac{181}{2},\ y=\frac{179}{2}$$

Thus, the first attempt itself does not result in x and y being natural numbers. But then the second way of writing 180 as a product of two natural numbers does give us a solution ...

$$180 = 2\times90 \quad \Rightarrow x+y=90,\ x-y=2 \quad \Rightarrow x=46,\ y=44$$

(Check: $46^2 - 44^2 = 2116 - 1936 = 180$)

Thus, few of the 9 ways of writing 180 as a 'product of two natural numbers' result in solutions of writing 180 as 'difference of squares' whereas few of them do not. How do we ascertain how many of the 9 ways would result in natural values of x and y?

When writing as difference of squares, why does number being even or odd matter?

In both these cases we are looking finding x and y where $(x + y) \times (x - y)$ = given number.

If number is odd, then both $(x + y)$ and $(x - y)$ are necessarily odd. Check all cases of earlier example. And if $(x + y)$ = odd and $(x - y)$ = odd, then $x = \frac{\text{odd}+\text{odd}}{2}$ and $y = \frac{\text{odd}-\text{odd}}{2}$ are both natural numbers.

If number is even, either both $(x + y)$ and $(x - y)$ could be even OR one of them could be even and one odd. When one is even and one odd, we get $x, y = \frac{\text{even}\pm\text{odd}}{2}$ and we would not get x and y as natural numbers. Only when both, $(x + y)$ and $(x - y)$ are even, would we get natural values for x and y.

Natural values for x and y would be obtained only when both $(x + y)$ and $(x - y)$ are even. Hence the first case (180×1) did not result in a solution whereas the second case (90×2) did give a solution for x and y.

Since both $(x + y)$ and $(x - y)$ are even, we could assume them as $2m$ and $2n$ respectively and thus we have $2m \times 2n = 180$ i.e. $m \times n = 45$. Each way of writing 45 as a 'product of two natural numbers' would result in writing 180 as a 'difference of squares'. Since 45 ($= 3^2 \times 5$) has 6 factors, it can be written as a product of two natural numbers in 3 ways. Thus 180 can be written as difference of squares in three ways.

$$45 = 1 \times 45 \quad \Rightarrow m = 45, n = 1 \quad \Rightarrow x + y = 90, x - y = 2 \quad \Rightarrow x = 46, y = 44$$

$$45 = 3 \times 15 \quad \Rightarrow m = 15, n = 3 \quad \Rightarrow x + y = 30, x - y = 6 \quad \Rightarrow x = 18, y = 12$$

(Check: $18^2 - 12^2 = 324 - 144 = 180$)

$$45 = 5 \times 9 \quad \Rightarrow m = 9, n = 5 \quad \Rightarrow x + y = 18, x - y = 10 \quad \Rightarrow x = 14, y = 4$$

(Check: $14^2 - 4^2 = 196 - 16 = 180$)

Procedure to write n as difference of two squares

If n is odd:

Total number of ways $= \dfrac{\text{Number of factors}}{2}$. (If number of factors is odd, then the number is itself a square and so one more way is $\left(\sqrt{n}\right)^2 - 0^2$ i.e. $n - 0$. Check if 0 is acceptable in the questions i.e. does question state squares of natural numbers or whole numbers.)

Each way of writing n as a product of squares will result in one way of writing as difference of squares.

If n is even:

Divide n by 4, say m. Total number of ways $= \dfrac{\text{Number of factors of } m}{2}$. (Again if number of factors is odd, n itself would be a square and one more way would be $n - 0$, counting of which is dependent if 0 is an acceptable square (does question state squares of natural numbers or of whole numbers).

Number of Integer solutions to $\frac{1}{a} \pm \frac{1}{b} = \frac{1}{k}$

Consider that we are searching for all integral solutions of x and y that satisfy $\frac{1}{x} + \frac{1}{y} = \frac{1}{12}$.

The given equation can be multiplied by $12xy$ to get rid of the fractions,

$12y + 12x = xy$

i.e. $xy - 12x - 12y = 0$

i.e. $(x - 12) \times (y - 12) = 144$. (Adding 144 to both sides)

And thus again we come to a scenario where we have to write 144 as a product of two integers, say a and b. Each of these would result in a unique integral solutions of $x = 12 + a$ any $y = 12 + b$

144 ($=2^4 \times 3^2$) has 15 factors and can be written as a product of two natural numbers in 8 ways (one way will be where both are equal to 12).

Considering order to be significant (i.e. $a = 1$ & $b = 144$ is different from $a = 144$ & $b = 1$), we would have 15 different solutions for a and b i.e. for x and y. (It won't be 16 because one solution is where both are 12 and the order in this case would not result in new solution)

In this example, also because x and y are Integers (and not Natural numbers), both could be positive or both could be negative. Thus, there would be a total of 30 solutions.

Lastly, one of the above 30 solutions, namely when both a and b are -12, results in $x = y = 0$ and this solution will not be accepted. Thus, final answer will be 29 ways.

E.g. 15: Find the number of solutions to $\frac{3}{x} - \frac{4}{y} = 1$ such that both x and y are natural numbers.

$\frac{3}{x} - \frac{4}{y} = 1 \quad \Rightarrow xy + 4x - 3y = 0$

$\Rightarrow (x-3)(y+4) = -12$ i.e. $a \times b = -12$, where $x = a + 3$ and $y = b - 4$.

For y to be a natural number, b has to be positive and greater than 4. Because b is positive, hence a has to be negative. But for x to be a natural number, a has to be negative but greater than -3.

Starting with a and taking negative values greater than -3,

$a = -1, b = 12 \quad \Rightarrow x = 2, y = 8 \quad \left(\text{Check} : \frac{3}{2} - \frac{4}{8} = 1.5 - 0.5 = 1\right)$

$a = -2, b = 6 \quad \Rightarrow x = 1, y = 2 \quad \left(\text{Check} : \frac{3}{1} - \frac{4}{2} = 3 - 2 = 1\right)$

Thus, there are two natural number solutions to $\frac{3}{x} - \frac{4}{y} = 1$.

Values of x for which $\frac{x+k}{x-k}$ is an Integer

E.g. 16: For how many integral values of x, would $\frac{x+36}{x-36}$ be an integer?

Say, $\frac{x+36}{x-36} = a$, where a is an integer. Subtracting 1 from both sides,

$\frac{x+36}{x-36} - 1 = a - 1 \quad \Rightarrow \frac{72}{x-36} = a - 1$

Since (a – 1) is also an integer, (x – 36) has to be a factor of 72. We know that 72 (=$2^3 \times 3^2$) has 12 factors, but in this case (x – 36) could be positives as well as negatives of the factors and thus could assume 24 values.

We actually do not need to find the values of x, but for those interested......

$x - 36 = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, ... \pm 24, \pm 36, \pm 72$

$\Rightarrow x = 37, 38, 39, 40, 42, ..., 72, 108 \;\; or \;\; 35, 34, 33, 32, 30, ..., 12, 0, -36$

Exercise

24. In how many ways can 144 be written as a product of two distinct integers?

1. 7 2. 8 3. 14 4. 15 5. 16

25. n is a composite number such that $\sqrt{n}$ is not a natural number. Choose the appropriate option choice regarding the following statements.

I. n has a factor between 1 and $\sqrt{n}$ II. n has a factor between $\sqrt{n}$ and n.

1. Both statements are true
2. Both statements are false
3. I is true and II is false
4. I is false and II is true
5. The statements could be true or false and cannot be ascertained uniquely.

26. All factors of 72^2 are written in increasing order, from left to right. Find the factor which occupies the 18th position from the left end.

1. 64 2. 72 3. 96 4. 120 5. 144

27. If all the factors of n be written in increasing order, the product of the 8th and the 18th factor is n. Find the number of factors of n.

1. 25 2. 26 3. 28 4. 30 5. 36

28. How many pairs of natural numbers are there whose difference of squares is 45?

1. 3 2. 4 3. 5 4. 6 5. 8

29. In how many ways can 945 be written as a difference of two squares of natural numbers?

1. 6 2. 8 3. 12 4. 16 5. 20

30. In how many ways can 288 be written as a difference of two squares of natural numbers?

1. 5 2. 6 3. 7 4. 8 5. 9

31. What is the product of all factors of 180?

1. 180^{18} 2. 180^{10} 3. 180^{9} 4. 180^{8} 5. None of these

32. Two natural numbers a and b are such that b is 72 more than a. For how many values of a, is the fraction $\frac{b}{a}$ an integral value?

1. 6 2. 8 3. 9 4. 12 5. 15

33. Find the number of natural number solution to $\frac{4}{x}+\frac{3}{y}=\frac{1}{12}$

1. 12 2. 14 3. 16 4. 24 5. 28

HCF & LCM

Highest Common Factor (or Greatest Common Divisor) of a set of numbers is the 'largest number that can completely divide all the given numbers' …

Highest ⇒ largest

Factor ⇒ completely divides

Common ⇒ all the numbers

Thus, if we need to find the largest number that can completely divide 72, 108 and 126, we are searching for the HCF of {72, 108, 126}.

Factors of a number are those that can completely divide the given number.

Factors of 72: 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.

Factors of 108: 1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108

Factors of 126: 1, 2, 3, 6, 7, 9, 14, 18, 21, 42, 63, 126

The Common Factors i.e. 1, 2, 3, 6, 9 and 18 are the ONLY numbers that can completely divide each of the three given numbers viz. 72, 108, 126. There is no other number that can divide each of the three numbers. E.g. both 72 and 108 are divisible by 36, but 126 is not.

Among these Common Factors, the highest is 18. Thus, we can safely say that 18 is the largest number that can divide each of 72, 108 and 126. There is no number higher than 18 that can divide the given numbers. So the Highest Common Factor of 72, 108 and 126 is 18.

Situations where HCF has to be used:

In situations when a quantity/number has to be divided into smaller equal groups, we would be making the use of factors. E.g.

- plant 120 saplings into rows having equal saplings
- measuring 16 litres using one smaller measuring can,
- dividing/cutting a rectangle of dimension 24 × 36 into equal squares without any wastage

In each of these cases, a factor will necessarily play a role –

- the number of saplings (or the number of rows) would have to be a factor of 120 (we cannot plant 25 saplings in a row as after 4th row we will be left with only 20 and the row will not have same number of saplings)
- the measuring can has to be a factor of 16. With 5 ltr measuring can we can only measure 5, 10, 15, 20, … and not 16 ltrs.
- the side of the square has to be factor of 24 and 36. Squares of side 8 cannot be formed as the side measuring 36 will not be completely utilized.

Finding the HCF

Approach 1: Factorisation

Factorise each of the given number. The HCF will be the product of the least powers of primes across the given numbers.

$72 = 2^3 \times 3^2$ $\quad$ $108 = 2^2 \times 3^3$ $\quad$ $126 = 2 \times 3^2 \times 7$

Thus, HCF = $2 \times 3^2 = 18$.

Obviously, this method is too text-bookish and should not be resorted to while find the HCF.

> Should we not take the highest power since we finding Highest C F?
>
> We are finding the 'highest' common factor. Then why do we take the 'least' power? Shouldn't we take the highest power?
>
> While we are finding the 'highest' common factor, more importantly, the required number should be a factor of all the numbers. If we take the highest power of a prime, 2^3 in this example, it would not divide 2 or 2^2. But the least power would definitely divide the higher powers.
>
> So the first condition is that it should divide all the given numbers and while satisfying this condition we have to find the largest number.

Approach 2: Highest power of primes that divide ALL the numbers

If the numbers are very easy, like the above example, 72, 108 and 126, just find the highest power of each primes (2, 3, 5, 7, ... taken turn by turn) that can divide each of the given numbers. Dividing each of the numbers will ensure a common factor and taking the highest power will ensure we get the largest such number.

Powers of 2: 72 can be divided by 2, 4 and 8; 108 can be divided by only 2 and 4; 126 only by 2. Thus HCF, being a 'common' factor, will have 2 and not 2^2 or 2^3 (else it would not be a factor of 126)

Powers of 3: 72 can be divided by 3 and 9; 108 by 3, 9 and 27; 126 by 3 and 9. Thus HCF will have 3^2.

> Remember divisibility rules:
>
> To check divisibility by 2, 4, 8, just check for last digit/last 2 digits/last 3 digits being divisible by 2, 4, 8 respectively
>
> To check divisibility by 3 or 9, just check for sum of digits being divisible by 3 or 9

Powers of higher primes: 5, 7 or higher primes do not divide atleast one of the given numbers. (72 can only be divided by 2 or 3 and no other prime). So they would not be present in the HCF.

Thus, HCF is 2×3^2.

E.g. Find the HCF of 180, 300 and 420

Checking for divisibility by 2, 4 and 8, we find that each of the number can be divided by 4.

Checking for divisibility by 3 and 9, while 180 is divisible by 9, 300 and 420 are divisible by only 3.

Also all the three numbers can be divided by 5.

No other prime number needs to be checked because 180 can be factorised into only multiples of 2, 3 and 5. Thus HCF = $4 \times 3 \times 5 = 60$.

Approach 3: Break numbers into manageable multiples

Think of the given numbers as a product of smaller numbers, anything that comes to your mind first (but avoid small numbers like 2, 3, 4, etc.) E.g. while finding HCF of 72, 108 and 126, one may think of:

$72 = 6 \times 12$ or 8×9 $\quad\quad$ $108 = 12 \times 9$ $\quad\quad$ $126 = 2 \times 63$

Since 63 is just 7×9, we just have to check if the other two numbers can be divided by any of 2, 7 and 9. Obviously, both numbers can be divided by 2 and 9 but not by 7. Thus, we can conclude that 2×9 is the highest factor common to all the three numbers.

E.g. Find the HCF of 180, 300 and 420

$180 = 18 \times 10$ $\quad\quad$ $300 = 30 \times 10$ $\quad\quad$ $420 = 42 \times 10$

Thus 10 is common to all the three numbers and from the rest {18, 30 and 42}, we can find 6 to be the highest common factor. Thus HCF = $10 \times 6 = 60$.

Approach 4: Finding the HCF of differences

When the numbers are not the regularly used ones and appear unwieldy, another great technique to find HCF could be to find the HCF of the differences.

Find the HCF of 238, 391 and 442.

Factorising or even finding a factor of 391 can be frustrating. In such cases, find the difference between pairs of numbers ...

$391 - 238 = 153$ $\quad\quad$ $442 - 391 = 51$.

The required HCF will be the HCF of these differences (51 and 153) or a factor of this HCF.

Obviously, the HCF of 51 and 153 is 51 (as 153 is a multiple of 51).

51 is not a factor of 238. Also since $51 = 17 \times 3$, one just needs to check that the 238, 391 and 442 are all divisible by 17 and the required HCF is 17.

Why does this work?

Consider h to be HCF of three numbers x, y, z. Thus each of x, y, z is a multiple of h i.e.

$x = h \times a$, $y = h \times b$ and $z = h \times c$

Taking the differences,

$(x - y) = h \times (a - b)$ and $(y - z) = h \times (b - c)$.

Thus, h is a definitely a common factor of the two differences $(x - y)$ and $(y - z)$.

(h need not be the HCF of $(x - y)$ and $(y - z)$ because there may also be a common factor in the other multiplicands viz. $(a - b)$ and $(b - c)$. But then h is the HCF of the differences or a common factor smaller than the HCF of the differences)

This funda can easily be used partly: the HCF of a set of numbers has to be a factor of the difference between any pair of numbers

Application of HCF:

Largest number 'that divides'/'on dividing' ......

E.g. 1: Find the largest number that on dividing 85, 150 and 220 leaves a remainder of 1, 6 and 4 respectively.

The number on dividing 85 leaves a remainder of 1. Thus the number can completely divide 84.

Similarly the number can completely divide 144 and 216. Thus we are looking for the largest number that completely divides 84, 144 and 216 i.e. HCF of 84, 144 and 216.

Now, $84 = 12 \times 7$. Its obvious that 7 does not divide the other number. So checking for 12, we find that each of the other numbers is divisible by 12. Thus HCF of 84, 144 and 216 (and the required answer) is 12.

Algebraically ....

While not necessary, one should be able to write the above logic algebraically as well. As seen in next example, it may be needed sometimes.

n on dividing 85 leaves remainder 1 $\Rightarrow 85 = n \times a + 1 \Rightarrow 84 = n \times a$

n on dividing 150 leaves remainder 6 $\Rightarrow 150 = n \times b + 6 \Rightarrow 144 = n \times b$

n on dividing 220 leaves remainder 4 $\Rightarrow 220 = n \times c + 4 \Rightarrow 216 = n \times c$

Thus n is a common factor of 84, 144 and 216 and we want to find the largest such number.

E.g. 2: Find the largest possible three digit number that leaves the same remainder when it divides 145, 277, 673.

Let's assume the number we want to find to be n and the common remainder in each case to be r. Thus we have,

$145 = n \times a + r$ $\quad$ $277 = n \times b + r$ $\quad$ $673 = n \times c + r$

Subtracting,

$132 = n \times (b - a)$ $\quad$ $396 = n \times (c - b)$ $\quad$ $528 = n \times (c - a)$

Thus n has to be a factor of 132, 396 and 528. Finding the HCF of these numbers, 132 = 11 × 12. Checking if the other two numbers are divisible by 11 and 12, we find that they are (just apply divisibility rules of 11, 4 and 3).

Thus, the largest value that n can take is 132.

(Check: 132 on dividing 145, 277 and 673 leaves a remainder of 13 in each case)

Grouping/dividing given quantities:

E.g. 3: 84 rose plants, 126 marigold plants and 189 chrysanthemum plants have to be planted in rows such that each row has equal number of plants and each row has plants of a particular variety only. What is the least number of rows needed?

Since the 84 rose plants have to be planted in rows having equal number of plants, the number of plants in each row has to be a factor of 84. i.e. there could be 7 plants in a row (& 12 rows) or 12 plants in a row (& 7 rows) or 14 plants in a row (& 6 rows) and so on. But a number that is not a factor of 84 cannot be the number of plants in a row. E.g. 10 plants can't be in a row because then after 8 complete rows, 4 plants will be leftover.

Similarly the number of marigold plants in a row has to be a factor of 126 and the number of chrysanthemum plants in a row has to be a factor of 189.

Since all rows should have same number of plants, the number of plants in a row has to be a common factor of 84, 126 and 189.

Also the number of rows will be least when each row has as many plants as possible, i.e. each row as plants equal to the HCF of 84, 126 and 189.

84 = 7 × 12.

A glance at 189 tells us that it is not divisible by 12, so checking divisibility by 7, we have

126 = 7 × 18 $\quad$ 189 = 7 × 27.

So obviously 7 is common to all three numbers and 12, 18 and 27 will yield another common factor i.e. 3.

Thus number of plants in each row = HCF of 84, 126 and 189 = 7 × 3 = 21.

Thus rows required will be $\frac{84}{21}+\frac{126}{21}+\frac{189}{21}$ i.e. 4 + 6 + 9 = 19 rows. We do not need to calculate this, we have already factorised 84, 126 and 189 and so use these factorised expressions to do the division.

When the HCFs are given and the numbers unknown

If HCF of x, y and z is given to be h, it should be obvious that h is a factor of each of the numbers x, y and z. Thus x, y and z should be multiples of h. So we can say that,

$x = h \times a$; $y = h \times b$; $z = h \times c$.

Further since h is the 'highest' common factor, a, b and c should not have any further common term. Or else that would also be common to the three numbers and the HCF would no longer be h but would become ($h \times$ that common term).

E.g. 4: How many pairs of numbers exists such that their HCF is 12 and the sum of the numbers is 108.

The numbers can be assumed as $12a$ and $12b$.

$12a + 12b = 108 \quad \Rightarrow a + b = 9$

Since the question is "how many pairs exists…", we need to see how many solutions exists for $a + b = 9$.

The following could be the values of a and b that satisfy the given condition

a : 1 2 3 4

b : 8 7 6 5

Post this we will get the same numbers in reverse order leading to the same pair of numbers $12a$ and $12b$.

But thinking that there are 4 such pairs is wrong. The values (3, 6) for a and b will lead to numbers 12×3 and 12×6. These pair of numbers will not have 12 as the HCF. The HCF would be 12×3.

Thus among the solutions we got, we have to check to see that a and b do not have any common factor. Thus, in this case, the answer would be 3 pairs.

Exercise

1. A milk man has to deliver 56 lts, 84 lts and 70 lts of milk to three different customers. What is the largest volume of measuring-can that he has to keep to measure the exact required quantity?

   1. 4   2. 7   3. 8   4. 12   5. 14

2. n on dividing 50, 75 and 125 leaves a remainder of 2, 3 and 5 respectively. What is the largest value that n can take?

   1. 12   2. 16   3. 20   4. 24   5. 32

3. A school has 120, 192 and 144 students enrolled for its science, arts and commerce courses. All students have to be seated in rooms for an exam such that each room has students of only the same course and also all rooms have equal number of students. What is the least number of rooms needed?

   1. 12   2. 19   3. 20   4. 24   5. 36

4. A rectangular cloth measuring 54 cm × 90 cm has to be cut into equal squares such that no cloth is wasted. What is the least number of squares that can be made?

   1. 6   2. 15   3. 16   4. 18   5. 19

5. The HCF of two numbers 3. If the sum of the two numbers is 36, how many such pair of numbers exists?

   1. 1   2. 2   3. 3   4. 4   5. 6

6. The HCF of two numbers is 24. If the product of the two numbers is 11520, how many such pair of numbers exists?

   1. 1   2. 2   3. 3   4. 6   5. 12

7. A number n leaves the same remainder while dividing 5814, 5430, 5958. What is the largest possible value of n?

   1. 144   2. 96   3. 72   4. 48   5. 24

8. In the above question (Q # 7), how many two digit values can n assume?

   1. 1   2. 2   3. 4   4. 8   5. 10

9. A cuboid of dimension 48 × 120 × 168 is cut into smaller equal cubes. What is the minimum number of cubes that can be cut?

   1. 12   2. 14   3. 24   4. 48   5. 70

10. If the HCF of 120 and n is 8, how many two digit values could n assume?

    1. None   2. 1   3. 2   4. 3   5. More than 3

LCM

Least Common Multiple of a set of numbers is the 'smallest number that can be completely divided by all the numbers in the given set'.

Least ⇒ smallest

Multiple ⇒ is divided by

Common ⇒ all the numbers

Thus, if we need to find the smallest number that can be completely divided by 6, 10 and 15, we are searching for the LCM of {6, 10, 15}.

Multiples of a number are those numbers that can be completely divided by the given number.

Numbers that are divisible by 6: 6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 72, ...

Numbers that are divisible by 10: 10, 20, 30, 40, 50, 60, 70, ...

Numbers that are divisible by 15: 15, 30, 45, 60, 75, ...

The Common Multiples i.e. 30, 60, ... are the ONLY numbers that can be completely divided by each of the three given numbers viz. 6, 10, 15.

Among these Common Multiples, the least is 30. Thus we can safely say that 30 is the least number that can be divided by each of 6, 10 and 15. There is no number less than 30 that can be divided by the given numbers. So the Least Common Multiple of 6, 10 and 15 is 30.

Situations where LCM has to be used:

In situations where there is a repetition of an event, we would most likely be using LCM. E.g.

- Clock striking after every 10 mins or bus leaving every 10 mins
- A traffic light/neon signboard being lit for 2 mins and then off for 1 mins
- A sprinter running round a circular track and completing a round in 8 mins

Each of the event will be repeated after an multiple of the number used

- the clock will strike/bus will leave at 10 mins, 20 mins, 30 mins, ... i.e. multiples of 10
- the complete cycle of on and off will end at 3 mins, 6 mins, 9 mins, ... i.e. multiples of 3
- the sprinter will be at the start point at 8 mins, 16 mins, 24 mins, ... i.e. multiples of 8.

Finding the LCM:

Approach 1:Factorisation

Find the LCM of 24, 40 and 54.

Factorising, $24 = 2^3 \times 3$ $40 = 2^3 \times 5$ $54 = 2 \times 3^3$

Since we are looking for a multiple of each of the following, we would have to take the highest power of each of the primes present in the above factorised forms.

Thus the LCM = $2^3 \times 3^3 \times 5$

Should we not take the least power since we are finding the Least C M?

We are finding the 'least' common multiple. Then why do we take the 'highest' power? Shouldn't we take the least power?

While we are finding the 'least' common multiple, more importantly, the required number should be a multiple of all the numbers. If we take the least power of a prime, 2^1 in this example, it would not be a multiple of 2^3. But the highest power would definitely be a multiple of all the numbers.

That is the reason we have to include 5 in the LCM even though it appears in only one number and not in the other two. Had we not included it, we would not get a multiple of 40.

So the first condition is that it should be a multiple of all the given numbers and while satisfying this condition we have to find the least such number.

Approach 2: Highest power of primes that appear in ANY of the numbers

Considering the above example of finding the LCM of 24, 40 and 54...

The highest power of 2 that appears in any of the numbers is 2^3 (appears in 24 and 40)

The highest power of 3 that appears in any of the numbers is 3^3 (appears in 54)

Also 5^1 appears in the number 40.

There are no further primes that are present in any of the numbers.

Thus LCM = $2^3 \times 3^3 \times 5$

When to use:

This approach is pretty useful if we want to find the LCM of 'many' numbers and the numbers are relatively small.

Find the LCM of 4, 5, 6, 7, 8, 9, 10, 12, 15, 16, 18, 20, 21 and 25.

The highest power of 2 that appears in any of the number is 2^4 (in 16).

The highest power of 3 that appears in any of the number is 3^2 (in 9 and 18)

The highest power of 5 that appears in any of the number is 5^2 (in 25).

The highest power of 7 that appears in any of the number is 7 (in 7 and 21).

Since no other prime number is present in any of the numbers, the LCM is $2^4 \times 3^2 \times 5 \times 7$.

Approach 3: Multiple of largest number

Start with the largest number and find which of its multiple is divisible by all.

Find the LCM of 15, 24, 40

The LCM has to be a multiple of 40 i.e. 40, 80, 120, 160, ... Consider each of these in sequence and check which is divisible by 15 and 24.

40 and 80 is not divisible by 15, 120 is divisible by 15 and also by 24. Thus LCM is 120.

Applications of LCM:

Finding numbers that on being divided by a, b, c, ... leave given remainders

Case 1: When remainders is same in all divisions

Find the series of numbers that when divided by 12, 18 and 30 leave a remainder of 5 in each case.

These types of problems are characterised by the remainder being same in each case. For such problems the series of numbers is the (LCM of divisors) × a + the common remainder, where a is any whole number 0, 1, 2, 3, ...

Logical Explanation: The LCM is the least number that is completely divisible by the divisors. If we add the required remainder to the LCM, this will always be left over as the remainder.

Thus, the series of numbers for this example are LCM (12, 18, 30) × a + 5 = $180a$ + 5.

The smallest such number is 5, second number is 185, third number is 365 and so on.

Algebraic Explanation:

Let n be the number when divided by 12, 18 and 30 leaves a remainder of 5 in each case. Algebraically,

$n = 12a + 5 \Rightarrow (n - 5) = 12a$

$n = 18b + 5 \Rightarrow (n - 5) = 18b$

$n = 30c + 5 \Rightarrow (n - 5) = 30c$

Thus $(n - 5)$ is a multiple of 12, 18 and 30 and the least such value is the LCM.

So, n = LCM (12, 18, 30) + 5

E.g. 5: n is a number that on being divided by 3, 4 and 6 leaves a remainder of 2 in each case.

i. Find the smallest three digit value that n can assume

ii. Find the largest three digit value that n can assume

iii. Find the value just greater than 5555 that n can assume

iv. How many distinct values can n assume if $2000 < n < 8000$

Since the remainder is constant for all the divisors, n is of the form $12a + 2$.

i. Smallest three digit number is 100 and so we have to select a suitable multiple of 12 such that we 'just' cross 99. Obviously the required multiple of 12 is 108 and the answer is 110.

ii. Largest three digit number is 999 and we have to search for a multiple of 12 just less than 1000. Working backwards, 999 is not divisible by 2; 998 is not divisible by 3; 997 not by 2; 996 is divisible by 4 and 3 and thus also 12.

Thus, answer is 996 + 2 = 998.

iii. Dividing 5555 by 12, we find the remainder to be 11. Thus 5556 will be fully divisible by 12 and the required answer is 5556 + 2 = 5558.

iv. The number just larger than 2000 and also divisible by 3 and 4 is 2004 i.e. 12 × 167.

The number just less than 8000 and divisible by 3 and 4 is 7992 (just check divisibility by 3 for numbers in steps of 4 i.e. 8000, 7996, 7992). Also 7992 is 12 × 666.

All numbers of the type $12a + 2$ with a being 167, 168, 169, ……, 666 would satisfy the given condition. There are 666 – 167 + 1 i.e. 500 such numbers.

Alternately after finding the smallest number as 2004 and largest as 7992, and knowing that required numbers increment in steps of 12, the number of values that n could assume is $\frac{7992-2004}{12}+1 = \frac{5988}{12}+1 = 499 + 1 =$ 500.

Case 2: When the difference between divisors and respective remainders is a constant

Find the series of numbers that when divided by 12, 18 and 30 leave a remainder of 5, 11 and 23 respectively.

These types of problems are characterized by the fact that the difference between the divisors and remainders is a constant. In this case $12 - 5 = 18 - 11 = 30 - 23 = 7$. For such cases the series of numbers is the (LCM of divisors) × a – the constant difference, where a is any natural number 1, 2, 3, ...

Logical Explanation: The given number, say n, leaves a remainder of 5, when divided by 12. Thus if we add 7 to the number, the remainder of 5 and this extra 7 will make $(n + 7)$ fully divisible by 12. Since deficit, for complete division, in case of being divided by each of 18 and 30 is also 7, adding 7 will make $(n + 7)$ also divisible by 18 and 30. Thus $(n + 7)$ is a multiple of 12, 18 and 30 and so n is multiple of 12, 18 and 30 minus 7.

Thus, the series of numbers in this case is $180a - 7$. The smallest such number is 173, second smallest number is 353 and so on.

Algebraic Explanation:

Let n be the number that when divided by 12, 18 and 30 leaves a remainder of 5, 11 and 23. Algebraically,

$n = 12a + 5 \Rightarrow (n + 7) = 12(a + 1)$ (adding 7 to both sides)

$n = 18b + 11 \Rightarrow (n + 7) = 18(b + 1)$

$n = 30c + 23 \Rightarrow (n + 7) = 30(c + 1)$

Thus $(n + 7)$ is a multiple of 12, 18 and 30 and the least such value is the LCM.

So, n = LCM (12, 18, 30) – 7

E.g. 6: n is a number that on being divided by 3, 4 and 6 leaves a remainder of 1, 2 and 4 respectively.

i. Find the smallest four digit value that n can assume.

ii. Find the value closest to 700 that n can assume.

Since the difference between the divisors and remainders is a constant 2, the general form of n is $12a - 2$.

i. The number just larger than or equal to 1000 and which is divisible by 3 and 4 is 1008. Thus smallest four digit value that n could assume is $1008 - 2 = 1006$.

ii. The number on either side of 700 that is divisible by 12 i.e. by 3 and 4 is 696 and 708. Thus, the values of n could be 698 and 710, of which obviously 698 is closer to 700.

Case 3: General Case

Find the series of numbers that when divided by 8 and 11 will leave a remainder of 3 and 10 respectively.

This is a general case where nor are the remainders common and nor the difference between divisors and remainders a constant.

To tackle such general cases, start with the largest divisor and find numbers that satisfy the given condition. Thus, we are looking for numbers that when divided by 11 leave a remainder of 10, i.e. numbers of the type $11a + 10$. The series of such numbers is 10, 21, 32, 43, 54, ...

Take each of these numbers turn by turn and then check for the smallest of these that satisfies the other given condition/s.

In this example, none of 10, 21 or 32 when divided by 8 leaves a remainder of 3. But 43 when divided by 8 leaves a remainder of 3. Thus, 43 satisfies both the given conditions.

> **Why do we start with larger divisor?**
>
> Consider the first such number 43. We found it in the third iteration of multiple of 11. Had we started with 8, we would have reached it only on the fifth multiple of 8. Thus, with the larger number we move in larger steps and reach the first number faster.

Let's find the next number that satisfies the conditions or the series of such numbers.

43 when divided by 11 leaves a remainder of 10. If 11 or any multiple of 11 is added to 43, the sum when divided by 11 will still leave a remainder of 10 (the extra 11s added will be flushed out in the quotient). If any other number (other than a multiple of 11) is added, the remainder would not remain 10.

Similarly if we add any multiple of 8 to 43 and then if the sum is divided by 8, the remainder will still be 3.

So what number should be added to 43 so that when the sum is divided by 11 and 8 the remainders are still 10 and 3 respectively? It should be a multiple of 8 and 11 and the smallest such number is the LCM of 8 and 11.

The series of such numbers is (LCM of divisors) $\times\ a$ + first such number i.e. in this example the series of numbers is $88a + 43$.

Linkage with Arithmetic Progressions

The numbers that leave a remainder of 10 on being divided by 11 is the series

10, 21, 32, 43, ...i.e. an A.P. with common difference 11.

Similarly, numbers that leave a remainder of 3 on being divided by 8 is the series

3, 11, 19, 27, 35, 43, ... i.e. an Arithmetic Progression with common difference 8.

We are looking for numbers that satisfy both the conditions i.e. for terms that are common to both the series. Such numbers have been found to be

43, 131, 219, ... i.e. again an Arithmetic Progression with common difference 88, the LCM of the two common differences, 8 and 11.

E.g. 7: Find the largest three digit number that when divided by 15 and 18 leaves a remainder of 3 and 6 respectively.

Numbers that when divided by 18 leave a remainder of 6 are 6, 24, 42, 60, 78, ... The smallest of these numbers that when divided by 15 leaves a remainder of 3 is 78. So the general form of the required numbers is LCM (15, 18) × a + 78 i.e. $90a + 78$. Now the largest three digit number of this form is when $a = 10$ and the required answer is 978.

Exercise

11. If the LCM of $2^3 \times 3^2$ and $2^a \times 3^b$ is $2^3 \times 3^3$, how many different values can each of a and b take?

    1. 3, 2 2. 4, 3 3. 3, 1 4. 4, 1 5. 2, 1

12. n is number that when divided by 5, 6 and 8 leave a remainder of 3 in each case. How many distinct three digit values can n assume?

    1. 12 2. 10 3. 9 4. 8 5. 6

13. If n is the largest three digit number that on being divided by 6, 10 and 15 leaves a remainder of 5 in each case, find the sum of the digits of n.

    1. 27 2. 24 3. 23 4. 21 5. 18

14. When marbles from a bag were divided into groups of 8 marbles, three marbles were left; when groups of 10 were made, again three marbles were left; and when groups of 12 marbles were made, again three marbles were left out. If in each grouping, atleast one group was formed, what is the least number of marbles that the bag contained?

    1. 3 2. 43 3. 63 4. 83 5. 123

15. n is a number that when divided by 3, 4 and 6 leaves a remainder of 1, 2 and 4 respectively. Find the number closest to 500 that n can take.

    1. 490 2. 492 3. 502 4. 504 5. None of these

16. A number when divided by 4, 5, 6, 7, 8, 9 and 10 leaves a remainder of 3, 4, 5, 6, 7, 8 and 9 respectively. What is the smallest such number?

    1. 2519 2. 2539 3. 2559 4. 2579 5. 2599

17. n is a number that when divided by 12 and 15 leave a remainder of 2 and 8 respectively. How many distinct three digit numbers can n assume?

    1. 17 2. 16 3. 15 4. 14 5. 13

18. If n is the smallest number that on being divided by 4, 5 and 7 leaves a remainder of 3, 1 and 2 respectively, find the sum of digits of n?

    1. 5 2. 6 3. 8 4. 12 5. 15

19. If the LCM of 75 and n is 150, how many different values can n assume?

    1. 1 2. 2 3. 4 4. 5 5. 6

20. The LCM of two numbers is 80 and the product of the two numbers is 320. How many such possible pairs of numbers exist?

    1. None 2. 1 3. 2 4. 3 5. More than 3

HCF and LCM

Some General Observations

1. For two numbers (only) ......

   Product of two numbers = HCF × LCM

2. For any set of numbers, the HCF has to be a factor of the LCM

Since HCF is a factor of all the given numbers and LCM is a multiple of all the numbers, the HCF would also be a factor of the LCM or other way round, the LCM has to be a multiple of HCF.

Don't fall for the following fundamental error ....

E.g. 8: The HCF and LCM of two numbers is 45 and 300. If one of the

numbers is 225, find the other number.

If we blindly use point 1 above, the other number can be found as
$$\frac{45 \times 300}{225} = 60$$

But then the HCF of 60 and 225 is 15 and the LCM is 900 and not 45 and 300 (thevalues given in the questions).

There is nothing wrong with point 1 above. It remains valid for all cases of two numbers. In this case, the data itself is inconsistent – if the HCF of two numbers is 45, the LCM of the numbers has to be a multiple of 45. Since the given LCM, 300, is not, the situation is just not possible and the answer is 'not possible' or 'data inconsistent'.

3. The HCF of a set of numbers is equal to or less than the least number of the set. The LCM of a set of numbers is equal to or greater than the greatest number of the set.

This is obviously so because HCF, being a factor of all the numbers in the set, has to be a factor of the least number of the set as well. Thus it is equal to or lesser than the least.

And LCM, being a multiple of all the numbers in the set, has to be a multiple of the greatest number in the set as well. Thus it is equal to or greater than the greatest.

The following spatial relation should be kept in mind (with smaller numbers on the left and greater numbers on the right)

HCF $\{a < b < c < d < \ldots\ldots < p < q < r\}$ LCM

HCF divides all numbers.
HCF is less than or equal to a

All numbers divide LCM.
LCM is greater than or equal to r

HCF divides LCM

HCF being the smallest and LCM being the largest

When is the HCF equal to the smallest number of the set OR the LCM equal to the greatest number of the set? Would they occur simultaneously?

Whenever all the numbers of the set are multiples of the smallest number in the set, the HCF would be equal to the smallest number of the set. E.g. The HCF of 18, 36, 90 is 18 since both 36 and 90 are multiples of 18.

Whenever all the numbers of a set are factors of the largest number in the set, the LCM will be equal to the largest number of the set. E.g. The LCM of 6, 32, 48, 96 will be 96 as each of 6, 32, 48 are factors of 96.

Needless to say if HCF is the smallest number of the set it DOES NOT mean LCM is the largest number or vice-versa. Needless, because in the first example, while the HCF was the smallest of the set, the LCM is 180 (and not 90, the greatest of the set). And in the second example, while the LCM was the greatest of the set, the HCF is 2 (and not 6, the smallest of the set).

4. HCF & LCM of fractions

$$\text{HCF of fractions} = \frac{\text{HCF of numerators}}{\text{LCM of denominators}}$$

$$\text{LCM of fractions} = \frac{\text{LCM of numerators}}{\text{HCF of denominators}}$$

Note: The fractions have to be in the lowest reduced form.

Error prone area

In finding the HCF or LCM of fractions, make sure you reduce the fractions first. Else you are likely to get a wrong answer. And this is a very common error that paper setters capitalise on.

E.g. The HCF of $\frac{5}{3}, \frac{6}{4}, \frac{7}{5}$ is not $\frac{\text{HCF of } 5, 6, 7}{\text{LCM of } 3, 4, 5}$ i.e. $\frac{1}{60}$.

The fractions have to be first reduced i.e. $\frac{5}{3}, \frac{3}{2}, \frac{7}{5}$ and now the HCF is found as

$\frac{\text{HCF of } 5, 3, 7}{\text{LCM of } 3, 2, 5}$ i.e. $\frac{1}{30}$.

Remainders

Questions on this topic require one to find the remainder when any mathematical expression is divided by a divisor. E.g. find the remainder when 1237 × 1239 × 1241 is divided by 4; or find the remainder when 2^{96} is divided by 17; or find the remainder when $15^{23} + 23^{23}$ is divided by 19.

But first we shall start by understanding the division process and the meaning of remainders.

Understanding Division

We all know that when 63 is divided by 5, the quotient is 12 and the remainder is 3.

What we need to refresh is the meaning of this entire process of division, the interpretation of the quotient and more importantly that of the remainder.

Division is repetitive subtraction. From the dividend, 63 in this case, we keep subtracting the divisor, 5 in this case. The quotient is the number of times we can subtract and when subtracting is not possible any longer, the left-over is the remainder.

Our model

To start with we will consider any division as grouping of marbles.

Thus, dividing 63 by 5 would be considered as having 63 marbles on a table and then forming as many groups of 5 marbles as possible and taking them away. Whatever is left on the table in the end will be the remainder. The number of groups that can be taken away is the quotient, though we will never be interested in the quotient in this topic.

The example with our model would mean: when groups of 5 marbles are taken away from 63 marbles lying on a table, at the end 3 marbles will be left on the table top. Also we can take away 12 groups of 5 marbles each. Thus relation, 63 = 5 × 12 + 3 should be obvious

Possible values of the Remainder

With the above model, it should be obvious that if the divisor is n, the possible values for the remainder (the number of marbles left on the table after all possible groups of n marbles are taken away) has to necessarily be from 0 to $(n - 1)$. If n or more marbles were remaining, then one or more groups of n marbles could yet be taken away and finally the remaining marbles have to be less than n.

Thus, if the divisor is 5, the only possible values of the remainder are 0, 1, 2, 3 and 4; and if the divisor is 17, the possible values of the remainders are 0, 1, 2, 3, ……, 16.

Relation between Dividend, Divisor, Quotient & Remainder

We would also remember the relation: Dividend = Divisor × Quotient + Remainder.

And this is obvious, as seen in our model earlier,

Total number of marbles = Group size × Number of groups + Left-over on the table.

'a number when divided by d leaves a remainder r'.

We would very often come across the above expression. In such cases, using the above expression, the number could be assumed as $d \times a + r$, where a is the quotient and can be any whole number.

Remainders of basic mathematical operations

Mathematical operation here means addition, subtraction, multiplication or exponent (raised to positive integers powers only) of numbers. E.g. Each of the following means a mathematical operation

$1237 \times 1239 - 1241$; $482^2 - 239 + 10241^9$; $15^{23} + 23^{23}$

Let's say we want to find the remainder when any mathematical operation done on a, b, c, …, is divided by a given number. Our procedure would be to divide each of a, b, c, …, individually by the given divisor and find the remainders in each case say, r_a, r_b, r_c, …. The remainder when the entire mathematical operation is divided by the given number is same as the same mathematical operation done with r_a, r_b, r_c, … respectively. Thus,

if the dividend was $a + b - c$, then the remainder will be $r_a + r_b - r_c$;

if the dividend was $a \times b + c$, then the remainder will be $r_a \times r_b + r_c$;

if the dividend was $a^2 \times b^3 - c^4$, then the remainder will be $r_a^2 \times r_b^3 - r_c^4$

E.g. If we want to find the remainder when $1237 \times 1239 - 1241$ is divided by 4, we first find the remainder when each of 1237, 1239 and 1241 is divided by 4. In this case the remainders are 1, 3, 1 respectively. The remainder when the entire expression $1237 \times 1239 - 1241$ is divided by 4 will be the same operation done on the remainders i.e. $1 \times 3 - 1$ i.e. 2.

Similarly if we want to find the remainder when $482^2 - 239 + 10241^9$ is divided by 5, we first find the remainder when each of 482, 239 and 10241 is divided by 5 and then perform the same mathematical operation on the remainders as that performed on these numbers. Thus, the required remainder in this case is
$2^2 - 4 + 1^9$ i.e. 1

Explanation using our model

Why does the above work? Any complex mathematical expression can be broken into numerous basic operations of addition and subtraction of two numbers (multiplication is successive addition and exponents are successive multiplication). So the reason of the above will be explained logically for addition and subtraction of two numbers....

Consider two numbers m and n that when divided by 7 leave a remainder of r_m and r_n respectively. Think of two tables, first one having m marbles and second one having n marbles. When all possible groups of 7 marbles are taken away from each of the tables independently, there are r_m and r_n marbles left on the two tables respectively.

When $(m + n)$ is divided by 7 it simply means that the marbles on both the table are taken collectively. Again as many groups of 7 marbles as could be formed from both the tables together, can again be taken away. And it should be obvious that the total remaining marbles now would be $r_m + r_n$.

When $(m - n)$ is divided by 7, it simply means that the number of groups of 7 that could be formed from second table will now be subtracted from the number of groups formed from the first table. And also the remainder of second table, r_n, will be subtracted from the remainder of first table, r_m. So lesser groups would be formed which affects the quotient. And the remainder will be $r_m - r_n$.

Consider $(m + m + m)$ when divided by 7. This simply means that now we have three similar tables as the first table. Thus, the number of groups that can be formed will be thrice as many and all the left-overs would now amount to $r_m + r_m + r_m$. (for those who are arguing that $r_m + r_m + r_m$ could possibly form another group, you are right on target and this point will be explained immediately next)

This logic can be extrapolated to multiplication also because $15 \times m$ is nothing but m added fifteen times i. e. $(m + m + m + \ ....)$

Algebraic Logic

m and n when divided by 7 leave a remainder of r_m and r_n respectively.

Thus, we could assume $m = 7a + r_m$ and $n = 7b + r_n$, where a and b are the quotients and could be any whole number.

Now consider the three basic operation, $m + n$; $m - n$; $m \times n$.

1. $m + n = 7a + r_m + 7b + r_n = 7(a + b) + r_m + r_n$

Obviously the right most expression is a multiple of 7 plus $(r_m + r_n)$. Thus $(m + n)$ when divided by 7 would leave a remainder of $(r_m + r_n)$

2. $m - n = 7a + r_m - (7b + r_n) = 7(a - b) + r_m - r_n$

Obviously the right most expression is a multiple of 7 plus $(r_m - r_n)$. Thus $(m - n)$ when divided by 7 would leave a remainder of $(r_m - r_n)$

3. $m \times n = (7a + r_m)(7b + r_n) = 49ab + 7(ar_n + br_m) + r_m \times r_n$

In the right most expression, the first two terms are multiples of 7. Thus $m \times n$ when divided by 7 would leave a remainder of be $r_m \times r_n$

4. Since an exponent with positive integral index is same as successive multiplication, the logic remains same for exponents as well. E.g. $m^3 = (7a + r_m)^3$. When the right hand side is expanded, all terms except $r_m{}^3$ will be multiples of 7. Thus, when m^3 is divided by 7 the remainder would be $r_m{}^3$.

However complex a mathematical relation may be, it will be made up of these basic operations in parts and hence the logic can be extended to any mathematical operation.

On getting a remainder greater than the divisor

While doing the above procedure, it is quite common to get the remainder to be more than the divisor. E.g. The remainder when $17^3 + 18^3$ is divided by 7 is $3^3 + 4^3$ i.e. 27 + 64 = 91. But when the divisor is 7, how can the remainder be 91?

91 is the total number of marbles that are collected from the individual left-overs of each table after performing the given mathematical operation on the tables.

Obviously since this is more than 7, we would again take away as many groups of 7 as possible from these and then find the final leftover i.e. we will divide 91 by 7 and find the remainder. In this case since 91 is completely divisibly by 7, the remainder is 0.

On getting a negative remainder

Sometimes, we would also get a negative remainder. E.g. the remainder when 1004 × 3001 – 5006 × 7005 is divided by 12 is 8 × 1 – 2 × 9 i.e. –10. (8, 1, 2 and 9 are the remainders when 1004, 3001, 5006 and 7005 are individually divided by 12)

What does a remainder of –10 mean?

A positive remainder meant the number of marbles left-over on the table. A negative remainder implies that we taken away more groups than physically possible and thus we are left over with a negative number of marbles on the table. Thus, we would have to add back groups to the table to get back to a positive remainder.

In this case, since we were dividing by 12, adding back one group i.e. 12 marbles to the table would mean the remainder is –10 + 12 i.e. 2.

Algebraic Interpretation

When dividing a number by 12, when we reach a remainder of – 10, we can assume,

Number = $12a + (-10)$

$= 12(a - 1) + 12 - 10 = 12(a - 1) + 2$

We have reduced the quotient by 1 (adding back 12 to the table, in our model) and thus the number is a multiple of 12 plus 2 i.e. when divided by 12 leaves a remainder of 2

Thus, whenever we have a negative remainder, we just add the divisor (or keep adding i.e. any multiple of the divisor) so that the remainder gets back to its possible range.

We would be arriving at a negative remainder quite often. So learn to convert a negative remainder to its actual possible range pretty quickly ....

With a divisor of 17, a remainder of –15 implies a remainder of 2 (on adding back a group of 17)

With a divisor of 23, a remainder of –12 implies a remainder of –12 + 23 i.e. 11

With a divisor of 6, a remainder of –20 implies a remainder of –20 + 6 + 6 + 6 + 6 i.e. 4

In the last example (divisor being 6), we have taken away from the table a really large number of groups that were not physically possible and hence we have to add back 4 groups of 6 back to the table before we get a positive remainder.

Negative remainder of –1

In the above examples, we arrived at a negative remainder and then we converted it to get a positive remainder, in the possible range of remainders.

Sometimes we can also make our job far easier by converting positive remainders to their equivalent negative remainders to ease our calculations. Usually in such cases a negative remainder of –1 helps us a lot.

The remainder is –1 when the positive remainder is exactly 1 less than the divisor i.e. we are 1 marble 'short' of forming another group.

With a divisor of 17, a remainder of 16 is equivalent to a remainder of –1 (after taking away one more group of 17 from 16 marbles, we would be left with – 1)

With a divisor of 8, a remainder of 7 is equivalent to a remainder of –1 (after taking away one more group of 8 from 7 marbles, we would be left with – 1).

The following example shows us how a remainder of –1 can save calculation

E.g. 1: What is the remainder when 16^4 is divided by 17?

Since 16 when divided by 17 leaves a remainder of 16 itself, coming to the conclusion that remainder is 16^4 is of no use as we would again have to divide 16^4 as it is far larger than 17. One way to go about it is that converting 16^4 as 256^2. When 256 is divided by 17, the remainder is 1 and thus the answer would be 1^2 i.e. 1.

We need not even have done this small conversion had we interpreted the remainder when 16 is divided by 17 as –1. In this case the answer would directly be $(-1)^4$ i.e. 1.

And if we had to find the remainder when 16^5 is divided by 17, we could again use –1 to find the answer as $(-1)^5$ i.e. –1. And finally convert the remainder back to its positive equivalent i.e. 16.

Cyclicity of remainders when $a^{1,2,3,...}$ is divided by b

Consider finding the remainder when 9^n is divided by 7, where n could be any whole number.

Let's successively take n as 1, 2, 3, ……

$9^1 \div 7$ leaves a remainder of 2

$9^2 \div 7$ leaves a remainder of 2^2 i.e. 4

$9^3 \div 7$ leaves a remainder of 2^3 i.e. 8. Since this is greater than 7, we divide again to get the remainder as 1.

$9^4 \div 7$ leaves a remainder of 2^4 i.e. 16 i.e. 2

$9^5 \div 7$ leaves a remainder of 2^5 i.e. 32 i.e. 4

$9^6 \div 7$ leaves a remainder of 2^6 i.e. 64 i.e. 1

$9^7 \div 7$ leaves a remainder of 2^7 i.e. 128 i.e. 2

Thus we see that as n successively assumes 1, 2, 3, 4, …… the remainders are 1, 2, 4, 1, 2, 4, 1, 2, ……

Thus the remainders show a cycle of 1, 2, 4. And we have learnt enough about cyclicity (in chapter on Cyclicity) to utilise this fact to find the remainder for any value of n. Remember the puzzle on cycles in that chapter? Here it is again …

Puzzle……

What is the 72nd term in the series *a*, *b*, *c*, *d*, *e*, *a*, *b*, *c*, *d*, *e*, *a*, *b*, *c*, *d*, *e*, *a*, *b*, ……?

The above should be a pretty easy puzzle!

The series consists of 5 terms viz. *a*, *b*, *c*, *d* and *e* that are repeated.

Thus the entire cycle (*a*, *b*, *c*, *d*, *e*) would be completed at the 70th term and the 71st and 72nd terms would respectively be *a* and *b*.

A simple way of doing the above is to divide 72 by 5 and find the remainder, 2 in this case. The required term will be the 2nd in the cycle.

If the above seems simple enough, the same funda will be used here……

Once we know the cycle, any term in the cycle can be found out. If we want to find, say, the 83rd term in the series: 1, 2, 4, 1, 2, 4, 1, 2, 4, ……, we would divide 83 by 3 and find the remainder, 2 in this case. The 83rd term in the cycle will be the 2nd term i.e. 2.

Thus 9^{83}, when divided by 7 will leave a remainder of 2.

The remainders when any number a, raised to successive powers 1, 2, 3, 4, …… is divided by b will always have a cycle of values being repeated. However the cycle could be very (very) long and the calculations involved in identifying the cycle could be cumbersome. Thus we will have to resort to one more working model to ease our calculations.

Consider finding the remainder when 28^n is divided by 15. Since 28 divided by 15 leaves a remainder of 13, hence 28^n will leave a remainder of 13^n.

Assuming n successively as 1, 2, 3, 4, ...

$13^1 \div 15$, remainder is 13

$13^2 \div 15$, remainder is 13^2 i.e. 169 i.e. 4 (on dividing again by 15)

$13^3 \div 15$, remainder is 13^3. This is where the calculations become cumbersome. Even if we find 13^3, in the next step we would have to find 13^4 and so we would continuously be working with very large numbers. However there is a way out.

13^3 can be considered as $13^2 \times 13$. Considering 13^2 and 13 as two numbers, we know that they when divided by 15 leave remainders 4 (found in previous step) and 13 (the number itself) respectively. Thus, their product i.e. 13^3, will leave a remainder of 4×13 i.e. 52 i.e. 7 (dividing again by 15)

Similarly when we want to find the remainder on dividing 13^4 by 15, consider 13^4 as $13^3 \times 13$, and individually when each multiplicand is divided by 15 the remainders are 7 (found in previous step) and 13; hence on dividing 13^4 by 15 the remainder will be 7×13 i.e. 91 i.e. 1.

Similarly 13^5 when divided by 15 will leave remainder 1×13 i.e. 13. (1: remainder of previous step)

13^6 when divided by 15 will leave remainder 13×13 i.e. 169 i.e. 4 (13: remainder of previous step)

13^7 when divided by 15 will leave remainder 4×13 i.e. 52 i.e. 7 (4: remainder of previous step)

Thus, the remainders are 13, 4, 7, 1, 13, 4, 7, 1, 13, ...

Notation:

The above will be denoted as follows in the rest of this topic...

When 13^n is divided by 15, for successive values of n the remainders will be

$$\begin{array}{lccccccccc} n: & 1 & & 2 & & 3 & & 4 & & 5 \\ \text{Remainder}: & 13 & & \cancel{169}, 4 & & \cancel{52}, 7 & & \cancel{91}, 1 & & 13 \\ & & \overrightarrow{\times 13} & & \overrightarrow{\times 13} & & \overrightarrow{\times 13} & & \overrightarrow{\times 13} & \end{array}$$

The cycle will be shown once, i.e. till the first remainder is again arrived at. From here on the same calculations will again repeat. Further on, the top-most row depicting the value of n would also be done away with.

Once the cycle is known we can find the remainder when 13^n is divided by 15 for any value of n. E.g. if we want to find the remainder when 13^{75} is divided by 15, we just use the fact that the remainder will be the 75^{th} term in the series 13, 4, 7, 1, 13, 4, 7, 1, 13, ... Since the cycle is of 4 terms, on dividing 75 by 4 the remainder is 3, which means that the 75^{th} term in the cycle will be same as 3^{rd} term i.e. 7.

E.g. 2: What is the remainder when 58^{75} is divided by 11?

Since 58 divided by 11 leaves a remainder of 3, the remainder will be 3^{75}. However since this is a number larger than 11, we have to divide it again by 11 and find the remainder.

When 3^n is divided by 11, for successive values of n, the remainders will be:

$$
\begin{array}{lcccccc}
n: & 1 & 2 & 3 & 4 & 5 & 6 \\
\text{Remainder}: & 3 \xrightarrow{\times 3} & 9 \xrightarrow{\times 3} & \cancel{27}, 5 \xrightarrow{\times 3} & \cancel{15}, 4 \xrightarrow{\times 3} & \cancel{12}, 1 \xrightarrow{\times 3} & 3
\end{array}
$$

Thus the remainders are in the cycle 3, 9, 5, 4, 1, 3, 9, ... i.e. cycles of 5 different values. Since we want to find the remainder when 3^{75} is divided, we want the 75th term and it's obvious that it will be the last in the cycle i.e. 1 (since 75 is multiple of 5 and cycle ends at 75th term).

E.g. 3: What is the remainder when 315^{315} is divided by 104

Since 315 when divided by 104 leaves a remainder of 3, the remainder when 315^{315} is divided by 104 will be 3^{315}. Since this is more than 104, we again have to divide 3^{315} with 104. Finding the cycle of remainders when 3^n is divided by 104 for successive values of n,

$$
\text{Remainders}: 3 \xrightarrow{\times 3} 9 \xrightarrow{\times 3} 27 \xrightarrow{\times 3} 81 \xrightarrow{\times 3} \cancel{243}, 35 \xrightarrow{\times 3} \cancel{105}, 1 \xrightarrow{\times 3} 3
$$

Thus, the remainders form a cycle of 6 different values. The 315th term in this cycle, will be the term equal to the remainder when 315 is divided by 6 i.e. the 3rd term i.e. 27.

Significance of remainder 1

The above examples would make it apparent that whenever the remainder is 1, the cycle of remainders will start repeating all over again.

$$
\begin{array}{lccccc}
n: & 1 & 2 & \ldots\ldots\ldots & r & \\
\text{Remainders}: & a \xrightarrow{\times a} & ? \xrightarrow{\times a} & \ldots\ldots\ldots \xrightarrow{\times a} & 1 \xrightarrow{\times a} & a \xrightarrow{\times a}
\end{array}
$$

Thus if a^r divided by b leaves a remainder of 1, the cycle of remainders will be of r distinct values.

In e.g. 2 above since $3^5 \div 11$ gave a remainder 1, the cycle of remainders had 5 different values.

In e.g. 3 above since $3^6 \div 104$ gave a remainder 1, the cycle of remainders had 6 different values.

Use of negative remainders

Sometimes the cycle of the remainders may be pretty long. However in few of these cases, we can reduce our work by half, if we keep a watch for remainder of –1 as well ......

What is the remainder when 2^{123} is divided by 17?

For successive values of n, the remainders when 2^n is divided by 17 are

$$\begin{array}{lccccccccccccccc}
n: & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 \\
\text{Remainders}: & 2 & & 4 & & 8 & & 16 & & \cancel{32}, 15 & & \cancel{30}, 13 & & \cancel{26}, 9 & & \cancel{18}, 1 \\
 & & \overrightarrow{\times 2} & & \overrightarrow{\times 2} & & \overrightarrow{\times 2} & & \overrightarrow{\times 2} & & \overrightarrow{\times 2} & & \overrightarrow{\times 2} & & \overrightarrow{\times 2} &
\end{array}$$

And after this the cycle of remainders will start repeating starting with 2, 4, ... all over again. Thus the cycle of remainders would be of 8 different values.

However if we would have noticed that 2^4 divided by 17 leaves a remainder of –1, we could have stopped just there and concluded that $2^4 \times 2^4$ i.e. 2^8 will leave a remainder of (–1) × (–1) i.e. 1. Thus the above work would be exactly halved

$$\begin{array}{lccccccc}
n: & 1 & & 2 & & 3 & & 4 \\
\text{Remainders}: & 2 & & 4 & & 8 & & \cancel{16}, -1 \\
 & & \overrightarrow{\times 2} & & \overrightarrow{\times 2} & & \overrightarrow{\times 2} &
\end{array}$$

To complete the question, the remainders when 2^n is divided by 17 are in cycles of 8 different values. And we need to find the 123^{rd} term of this cycle. The 123^{rd} term will be the term equal to the remainder when 123 is divided by 8 i.e. 3^{rd} term i.e. 8.

Thus when 2^{123} is divided by 17, the remainder is 8.

E.g. 4: Find the remainder when 11^{154} is divided by 13

For successive values of n, the remainders when 11^n is divided by 13 are

$$\begin{array}{lccccccccccc}
\text{Remainders}: & 11 & & \cancel{121}, 4 & & \cancel{44}, 5 & & \cancel{55}, 3 & & \cancel{33}, 7 & & \cancel{77}, 12, -1 \\
 & & \overrightarrow{\times 11} & & \overrightarrow{\times 11} & & \overrightarrow{\times 11} & & \overrightarrow{\times 11} & & \overrightarrow{\times 11} &
\end{array}$$

Since 11^6 when divided by 13 leaves a remainder of –1, 11^{12} when divided by 13 will leave a remainder of 1. Thus when 11^n is divided by 13, the remainders will have a cycle of 12 distinct values of which the first six we have already found to be 11, 4, 5, 3, 7, 12

Since we need the 154^{th} term in a cycle of 12 distinct values, we know that the required term will be 10^{th} term (154 ÷ 12, rem=10) of the cycle. But now, we do not know the 10^{th} term in the cycle, we just know the first 6 terms. So how do we identify the remainder?

We need to know the remainder when 11^{10} is divided by 13. Considering 11^{10} as $11^6 \times 11^4$ (since we know the remainders when 11^6 and 11^4 individually are divided by 13) we can find the remainder to be $(-1) \times 3$ i.e. -3 i.e. $-3 + 13 = 10$.

Thus, when 11^{154} is divided by 13, the remainder is 10.

(One could have factorised 11^{10} anywhich way, even as $\left(11^5\right)^2$ and found the remainder to be 7^2 i.e. 49 i.e. 10, on dividing again by 13)

Exercise

Find the remainder in case of each of the following division

1. $80^{81} \div 9$

   1. 0 2. 1 3. 4 4. 5 5. 8

2. $81^{81} \div 13$

   1. 1 2. 2 3. 3 4. 11 5. 12

3. $60^{60} \div 11$

   1. 1 2. 3 3. 5 4. 9 5. 10

4. $4^{33} \div 27$

   1. 1 2. 4 3. 13 4. 19 5. 26

5. $83^{1002} \div 39$

   1. 1 2. 5 3. 8 4. 25 5. 38

6. $9103^{220} \div 91$

   1. 1 2. 3 3. 9 4. 27 5. 81

7. $60^{60} \div 17$

   1. 1 2. 9 3. 13 4. 15 5. 16

8. $103^{101} \div 19$

1. 1 2. 7 3. 8 4. 12 5. 18

9. $3^{52} \div 244$

1. 3 2. 9 3. 27 4. 81 5. 243

10. $1000^{1000} \div 77$

1. 1 2. 2 3. 33 4. 44 5. 76

11. $110^{220} \div 21$

1. 1 2. 4 3. 5 4. 16 5. 20

12. $2^{99} \div 25$

1. 1 2. 12 3. 13 4. 15 5. 24

13. $7^{109} \div 17$

1. 16 2. 15 3. 11 4. 6 5. None of these

14. $(222^{333} + 333^{222}) \div 11$

1. 0 2. 1 3. 6 4. 7 5. 10

15. $(37^{64} - 27^{64}) \div 64$

1. 0 2. 1 3. 16 4. 32 5. 63

Composite Index

E.g. 5: What is the remainder when $16^{17^{18}}$ is divided by 7?

Note:

$16^{17^{18}}$ and $\left(16^{17}\right)^{18}$ are not the same.

$\left(16^{17}\right)^{18} = 16^{17 \times 18} = 16^{306}$. And in this case the problem could be solved using the methods explained earlier.

But in the case of $16^{17^{18}}$, the index of 16 is 17^{18}, which is a large number beyond scope of calculation.

Ignoring the index, 16^n when divided by 7 will leave a remainder of 2^n. And for large values of n, this would be larger than 7 and would again have to be divided by 7.

Since the number, 2, is small, for successive values of n being 1, 2, 3, ..., the remainder when divided by 7 should orally be found as 2, 4, 1, 2, ... That is the remainders have a cycle of three different values. Now we want to find the 17^{18}th term in this series. The 17^{18}th term will be the term equal to the remainder when 17^{18} is divided by 3.

17 when divided by 3 leaves a remainder of 2 i.e. (–1) and so 17^{18} will leave a remainder of $(-1)^{18}$ i.e. 1.

Thus the 17^{18}th term in the series 2, 4, 1, 2, 4, 1, ... will be same as the 1st term i.e. 2.

E.g. 6: What is the remainder when $69^{69^{69}}$ is divided by 11?

69^n when divided by 11 will leave a remainder of 3^n.

3^n when divided by 11, for successive values of n being 1, 2, 3, ..., will leave remainders 3, 9, 5, 4, 1, 3, ... i.e. in cycles of 5 distinct values.

We need to find the 69^{69}th term of this series, which will be the term equal to the remainder when 69^{69} is divided by 5.

69 divided by 5 leaves a remainder of (–1) and hence 69^{69} when divided by 5 will leave a remainder of $(-1)^{69}$ i.e. –1 i.e. 4.

Thus the required remainder will be the 4th term of the series i.e. 4.

E.g. 7: What is the remainder when $31^{31^{31}}$ is divided by 7?

31^n when divided by 7 will leave a remainder of 3^n.

3^n when divided by 7, for successive values of n being 1, 2, 3, …, will leave remainders 3, 2, 6 i.e. –1.

Since 3^3 divided by 7 leaves remainder –1, 3^6 when divided by 7 will leave remainder of 1.

Thus the cycle of remainders when 3^n is divided by 7 will have 6 distinct values. And we have to find the 31^{31}st term of this series, which will be the term equal to the remainder when 31^{31} is divided by 6.

31 divided by 6 leaves a remainder of 1 and hence 31^{31} when divided by 6 will also leave a remainder of 1.

Thus, the required remainder will be the 1st term of the series i.e. 3.

It is not always that we get a remainder of 1. In cases, the cycle would start repeating itself without yielding a remainder of 1. See the following two examples

E.g. 8: What is the remainder when 2^{117} is divided by 44?

When 2^n is divided by 44, the remainders for successive values of n being 1, 2, 3, …. are:

$$2 \quad 4 \quad 8 \quad 16 \quad 32 \quad \substack{\cancel{64} \\ 20} \quad 40 \quad \substack{\cancel{80} \\ 36} \quad \substack{\cancel{72} \\ 28} \quad \substack{\cancel{56} \\ 12} \quad 24 \quad \substack{\cancel{48} \\ 4} \quad 8 \quad 16$$

Thus, the same values again start repeating.

However please note that the first 2 does not belong to the cycle. Excluding this 2, there are 10 distinct values in the cycle. We need to find the 117th term of the cycle including the first 2 i.e. the 116th term of the cycle of 10 values excluding the 2. Since 116 ÷ 10 leaves a remainder of 6, the required value will be the 6th term in the series 4, 8, 16, …. Thus the answer is 40

ALTERNATE METHOD:

Without getting a remainder of 1, the cycle would start repeating itself in cases when there is a common factor between the dividend and the divisor. In this case 2^{117} and 44 have a common factor, 4.

Another way to handle such question is to cancel out the common factor first. However when the divisor is changed, one needs to adjust for this manipulation at the end.

When the divisor is reduced

What is the remainder when 80 is divided by 50?

Quite a lot of students are inclined to reduce $\frac{80}{50}$ to $\frac{8}{5}$ and find the answer as 3.

This is not correct, and obviously something should be wrong because when we divide by 50, the possible remainders are from 0 to 49 but when dividing by 5, the possible remainders are only 0, 1, 2, 3 and 4.

Since we have divided both numerator and denominator by 10, we would have to multiply the so found remainder, 3, back by 10 to get the correct answer as 30.

Obviously when 80 marbles are divided into groups of 50 marbles, one group is possible and the remainder will be 30.

Consider we divide both 80 and 50 by 5. Thus we are now dividing 16 by 10 and the remainder is 6. To arrive at the remainder when 80 is divided by 50, we need to multiply this 6 back with the 5 i.e. correct answer is 30.

Cancelling out 4 from 2^{117} and 44, we are left with 2^{115} divided by 11.

2^n when divided by 11, for successive values of n being 1, 2, 3, ... leave a remainder of 2, 4, 8, 5, 10 i.e. –1.

Thus, when divided by 11, 2^5 leaves a remainder of (–1) and 2^{10} will leave a remainder of 1.

Thus the cycle of remainder when 2^n is divided by 11 will have 10 distinct values repeating themselves and we need to find the 115th term of this series, i.e. the 5th term (115 ÷ 10, rem = 5) of this series. The 5th term of the series is 10.

Now 2^{115} divided by 11 will leave a remainder of 10. Multiplying back with 4, 2^{117} when divided by 44 will leave a remainder of 10 × 4 = 40.

Algebraic interpretation of dividing by 4

Consider that the remainder when 2^{117} is divided by 44 to be r. Thus, we have the relation

$2^{117} = 44 \times q + r$, where q is the quotient.

Dividing both sides of this equation by 4, we get

$2^{115} = 11 \times q + r/4$

Thus, we have found $r/4$ to be equal to 10 and thus r will be 40.

The above method can also be used in cases when the divisor is a very large number that can be reduced by cancelling out with the dividend ......

E.g. 9: What is the remainder when 17^{100} is divided by 153?

Noticing that 153 has 17 as a factor, dividing both dividend and divisor by 17, we now have to find the remainder when 17^{99} is divided by 9.

17 divided by 9 leaves a remainder of (–1) and hence 17^{99} when divided by 9 will also leave a remainder of (–1) i.e. 8.

However 8 is not the final answer as we had divided both dividend and divisor by 17. Thus the final answer will be 8 × 17 i.e. 136.

Exercise

Directions for question 16 to 25: Find the remainder in the case of each of the following division.

16. $13^{14^{15}} \div 7$

1. 1 2. 2 3. 3 4. 4 5. 6

17. $35^{36^{37}} \div 11$

1. 1 2. 2 3. 5 4. 9 5. 10

18. $35^{37^{39}} \div 11$

1. 1 2. 4 3. 8 4. 9 5. 10

19. $57^{67^{77}} \div 17$

1. 1 2. 6 3. 9 4. 12 5. 16

20. $4^{74} \div 6$

1. 0 2. 1 3. 2 4. 4 5. 5

21. $6^{25} \div 4$

1. 0 2. 1 3. 2 4. 3

22. $3^{333} \div 108$

1. 1 2. 3 3. 9 4. 27 5. 81

23. $17^{19^{21}} \div 51$

1. 1 2. 3 3. 17 4. 49 5. 50

24. $27^{37^{47}} \div 11$

1. 1 2. 2 3. 3 4. 4 5. 10

25. $25^{35^{75}} \div 15$

1. 0 2. 1 3. 5 4. 10 5. 14

26. What is the digit in ten's place of 2^{1001}?

1. 2 2. 4 3. 5 4. 6 5. 8

Miscellaneous Topics

Successive Division

When "A number n is successively divided by a and b", it means:

The number n is divided by a

The quotient obtained in the above division is then divided by b

Thus, in successive division, the quotient just obtained in the previous step is divided by the next divisor.

E.g.: What is the largest three digit number that on being divided "successively" by 5, 6 and 8 leaves a remainder of 1, 3 and 7 respectively?

Let the number be n. Since n when divided by 5 leaves a remainder of 1, we can say, $n = 5 \times a + 1$ ......*i*

In the above equation, a is the quotient. Because it is successive division, a is divided by 6 and the remainder is 3. Thus, $a = 6 \times b + 3$ ......*ii*

Next, b is divided by 8 and the remainder is 7. Hence, $b = 8 \times c + 7$ ......*iii*

Substituting the value of b from *iii* in *ii*, we have $a = 6 \times (8 \times c + 7) + 3$

$a = 6 \times 8 \times c + 45$

Substituting this value of a in *i*, $n = 5 \times (6 \times 8 \times c + 45) + 1$

$n = 5 \times 6 \times 8 \times c + 226$ ......*iv*

Thus there are many values of n that would satisfy the given conditions. If we keep substituting c with 0, 1, 2, 3, ... we would get the series of such numbers.

To find the largest three digit number of the form $240c + 226$, c would assume a value of 3 and the answer is 946

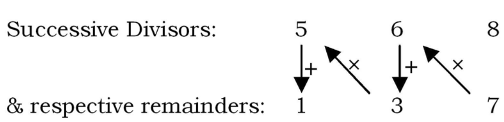
Short-cut

What needs to be learnt in the case of successive division is how can we arrive at the form as given in *iv* directly without going through the intermediate steps.

It should be obvious that the number is a multiple of the product of the divisors (and not the LCM) plus a constant. The constant in the earlier case was 226. So lets learn how to arrive at this value. If we see carefully, we have done the following to arrive at this number:

$5 \times (6 \times 7 + 3) + 1$

This can be memorized more easily by the following visual:



$7 \times 6 = 42$; $42 + 3 = 45$; $45 \times 5 = 225$; $225 + 1 = 226$

Consider an another example, say a number n is successively divided by 11, 12 and 15 and the remainders are 7, 3 and 10 respectively. One should be able to write the general form of the number n directly as follows:

$n = 11 \times 12 \times 15 \times a + 1360$

The way we got 1360 is: $10 \times 12 = 120$; $120 + 3 = 123$;

$123 \times 11 = 1353$; $1353 + 7 = 1360$.

Finding the number of times a digit appears

It would be a good idea to memorize that in any set of 100 consecutive natural numbers, say from 257 to 356, any particular digit, say 8 would be appearing 10 times in the unit's place (258, 268, 278, 288, ..., 338, 348) and 10 times in the ten's place (280, 281, 282, ..., 288, 289). Thus in all the 100 numbers 8 appears 20 times in the unit's and ten's place considered together. This is true of any digit and not necessarily just 8. Check this for yourself.

However there are only 19 numbers in any set of 100 consecutive natural numbers that contain the digit 8, or any other particular digit. This is because one number is common to the two sets of 10 occurrences of the digit (288 in the above example)

Further there are only 18 numbers in any set of 100 consecutive natural numbers which have a particular digit EXACTLY once.

E.g. 1: How many numbers from 283 to 839 have the digit 7?

283 to 882 is 600 consecutive natural numbers of which 100 of them are in seven hundreds, all of which have the digit 7. In the remaindering set of 500 numbers, the digit 7 would appear in $19 \times 5 = 95$ numbers. But then we have overshot the range given. From 840 to 882 there are 10 numbers having the digit 7 in the ten's place (870 to 879) and another 3 numbers having the digit 7 in the unit's place (847, 857, 867). Thus we have counted these 13 numbers also which we should not have as they are not in the given range. So our answer is $100 + 95 – 13 = 182$.

Alternately,

From 283 to 299, there are 2 numbers with the digit 7.

From 300 to 399, there are 19 numbers with the digit 7.

From 400 to 499, there are 19 numbers with the digit 7.

From 500 to 599, there are 19 numbers with the digit 7.

From 600 to 699, there are 19 numbers with the digit 7.

All of these could be clubbed as from 300 to 699 i.e. in 400 consecutive numbers there are $19 \times 4 = 76$ numbers having the digit 7 inthem.

From 700 to 799, there are 100 numbers with the digit 7.

From 800 to 839, there are 4 numbers with the digit 7.

Thus in all there are $2 + 76 + 100 + 4 = 182$ numbers with the digit 7.

E.g. 2: A person starts writes all four digit natural numbers. How many times has he written the digit 2?

1000 to 9999 are 9000 consecutive natural numbers. i.e. 90 sets of 100 consecutive numbers. In each set of 100 consecutive numbers the digit 2 will be written 20 times. So in 90 sets, the digit 2 will be written $90 \times 20 = 1800$ times.

But since these are 4 digit numbers, we also need to consider writing 2 in the hundred's and thousand's place as well.

Considering the hundred's position, the digit 2 will be written 100 times in each thousands i.e. in 1000's it will appear 100 times (from 1200 to 1299), in 2000's it will appear 100 times (from 2200 to 2299) and so on. Thus it will be written $100 \times 9 = 900$ times.

Finally, the digit 2 will be written in the thousand's place a total of 1000 times (2000 to 2999).

In all the digit 2 will be written $1800 + 900 + 1000 = 3700$ times.

Alternately with P & C

If one is comfortable with basic Permutations, one can solve this problem in a much more elegant way:

The digit 2 will appear in the unit's place a total of $9 \times 10 \times 10$ i.e. 900 times

Similarly the digit 2 will appear in each of ten's and hundred's place 900 times.

In thousands place the digit 2 will appear $10 \times 10 \times 10 = 1000$ times.

Thus in all the digit 2 will be written $1000 + 900 \times 3 = 3700$ times.

Alphametic

Alphametics (also called cryptarithms) are a particular type of puzzes where digits are represented by letters. Each letter represents a unique number all throughout the puzzle and each digit is represented by only one letter i.e. no digit can be represented by two different letters. To solve an alphametic is to find which letter stands for which digit.

E.g. 3: To start off with a very easy one...

$$\begin{array}{cccc} & & & I \\ + & & B & B \\ \hline & I & L & L \end{array}$$

One clue one can easily get is to look at the first digit (from the left) in the result, I in this case. I can be nothing other than 1, in this case, because the maximum sum of two digits can just be 18 (that too only if both are same). Even assuming there is a carry over of 1 from the previous column, the maximum sum of any column when two numbers are added could be 19. Thus $I = 1$.

Another clue is to find does carry over exists and where. In this case, from the addition of unit's place there has to be a carry over of 1 or else in the ten's place, in the result it should have been just B. Since $I = 1$ and $I + B$ has a carry over, the only possible solution is $B = 9$ and thus we also find out that $L = 0$. So the solution looks like:

$$\begin{array}{cccc} & & & 1 \\ + & & 9 & 9 \\ \hline & 1 & 0 & 0 \end{array}$$

E.g. 4: Now a difficult one...

$$\begin{array}{cccccccc} & & & S & T & E & E & L \\ + & & B & E & L & T & E & D \\ \hline & R & A & D & I & A & L & S \end{array}$$

By the logic explained, starting from the left end of the result one should easily deduce that $R = 1$, $B = 9$ and $A = 0$

Post this one would have to do some hit and trial. A good idea would be to start with E as it appears the most.

E cannot be 0 and 1 as they are already represented by other letters. Nor can it be 2 as then in the 5^{th} column from right, we can only get D to be 0 or 1. (Because S can at max be 8 and there has to be a carry-over from this fifth column to the sixth column).

Assuming $E = 3$, S has to be 8 and there has to be a carry-over of 1 from fourth column to fifth. Thus we get $D = 2$. Looking at the first column, $L + 2$ has to end in 8 and so $L = 6$. These values also satisfy the second

column with $E + E = L$ i.e. 3 + 3 = 6. Looking at third column from right, $E + T = 10$ (cannot be 0 or 20) and since E is assumed to be 3, $T = 7$.
The fourth column gives us $T + L + 1$ (carry-over from third column) i.e. 7 + 6 + 1 = 14 and $I = 4$. All values get satisfied and thus our assumption is right. Its not always that the assumption will be right in the first case itself, but its fast enough to work with assumptions.

Thus the answer in this case is:

$$\begin{array}{r} 8\ 7\ 3\ 3\ 6 \\ +\quad 9\ 3\ 6\ 7\ 3\ 2 \\ \hline 1\ 0\ 2\ 4\ 0\ 6\ 8 \end{array}$$

E.g. 5: This example deals with multiplication:

$$\begin{array}{r} C\ B\ A \\ \times\quad B\ A \\ \hline D\ C\ B\ A \end{array}$$

It should be obvious that A can only be 0, 1, 5 or 6 (or else it would not end with itself)

Also the last two digits of the product is dependent only on the last two digits of each multiplicand. So $BA \times BA$ should end with a BA. One would need to have a slight association with regular products and it should strike that squares of numbers ending with 25 always end with a 25. Thus working on an assumption of $A = 5$ and $B = 2$, the problem boils down to the following:

$$\begin{array}{r} C\ 2\ 5 \\ \times\quad 2\ 5 \\ \hline ?\ 2\ 5 \\ 5\ 0\ - \\ \hline D\ C\ 2\ 5 \end{array}$$

So we have the unit digit of $5 \times C + 1$ (carry-over) + 5 to be C. Trying different values 1, 3, 4, 6, … so on turn by turn, we see that C = 1 itself satisfies the condition and with $C = 1$, we get $D = 3$. So the solution is:

$$\begin{array}{r} 1\ 2\ 5 \\ \times\quad 2\ 5 \\ \hline 3\ 1\ 2\ 5 \end{array}$$

Surds

Surds are expressions involving irrational numbers like $\sqrt[m]{n}$. While there is hardly any stand alone question on surds, they may find an appearance in a question on any topic and thus one should be aware of how to handle them.

Rationalising a surd

One of the common techniques to handle expressions like $\frac{\text{any expression}}{\sqrt{a} \pm \sqrt{b}}$ is the process of rationalizing the denominator i.e. converting the denominator into a rational number.

Consider the expression $\frac{1}{\sqrt{3}-1}$. To rationalize the denominator, we multiply both the numerator and denominator with the conjugate pair of the denominator, $\sqrt{3}+1$ in this case ...

$$\frac{1}{\sqrt{3}-1} = \frac{1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{\sqrt{3}+1}{3-1} = \frac{\sqrt{3}+1}{2}$$

E.g. : Find the value of $\frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + ... + \frac{1}{\sqrt{20}+\sqrt{19}}$

Rationalising each denominator, the given expression is

$$\frac{\sqrt{2}-1}{2-1} + \frac{\sqrt{3}-\sqrt{2}}{3-2} + \frac{\sqrt{4}-\sqrt{3}}{4-3} + ... + \frac{\sqrt{20}-\sqrt{19}}{20-19}$$

$$= \left(\sqrt{2}-1\right) + \left(\sqrt{3}-\sqrt{2}\right) + \left(\sqrt{4}-\sqrt{3}\right) + ... + \left(\sqrt{20}-\sqrt{19}\right) \quad = \sqrt{20}-1$$

Completing the square

Another technique to handle surds in specific instances is by completing the squares. The following example shows the use of this technique...

E.g. : $\sqrt{13+5\sqrt{5}+\sqrt{9+3\sqrt{5}-\sqrt{14+6\sqrt{5}}}} = ?$

Look at the innermost root, $14+6\sqrt{5}$. Consider this to be the expansion $a^2 + 2ab + b^2$ with $6\sqrt{5}$ being the middle term. Thus, a = 3 and b = $\sqrt{5}$. Now we can write,

$$\sqrt{14+6\sqrt{5}} = \sqrt{9+2\times3\times\sqrt{5}+5} = \sqrt{\left(3+\sqrt{5}\right)^2} = 3+\sqrt{5}$$

The given expression will now become, $\sqrt{13+5\sqrt{5}+\sqrt{9+3\sqrt{5}-\left(3+\sqrt{5}\right)}}$ i.e.

$\sqrt{13+5\sqrt{5}+\sqrt{6+2\sqrt{5}}}$

Writing $6+2\sqrt{5}=1+2\times1\times\sqrt{5}+5=\left(1+\sqrt{5}\right)^2$, the given expression will become

$\sqrt{13+5\sqrt{5}+\left(1+\sqrt{5}\right)}=\sqrt{14+6\sqrt{5}}$

Writing the expression under the root as a square,

$\sqrt{14+6\sqrt{5}}=\sqrt{9+2\times3\times\sqrt{5}+5}=\sqrt{\left(3+\sqrt{5}\right)^2}$, the given expression simplifies to $3+\sqrt{5}$.

Digits of a number

There are several instances when you would have to assume the digits of a two digit number as x and y. If x is assumed as the digit in ten's place and y as the digit in unit's place, the number would "appear" like xy. But while forming equations, the number would have to be represented as $10x + y$ and not xy.

A blunder on considering number as xy

If the equation being formed is for 'ten's digit is added to the number', then the expression is NOT $x + xy$ but IS $x + (10x + y)$.

With $x + xy$, one would easily mistake xy as $x \times y$ and then take x common and manipulate the expression as $x \times (1 + y)$.

This is blatantly wrong because if the number were to be 53, adding ten's digit to the number results in 5 + 53 = 58. But $x \times (1 + y)$ will become $5 \times (1 + 3) = 20$.

If one is not clear about this, consider the number 47. The value of this number is NOT FOUR SEVEN, but is FORTY SEVEN i.e. 40 + 7. This comes from the positional value in any number, the 4 in the ten's place adds a value of 40 and not 4.

Consider an example, though very easy...

A two digit number when subtracted from the number formed by reversing the digits results in 72. Find the difference between the digits of the number.

Assuming the two digit number as xy, the number formed by reversing the digits is yx.

Now if we form the equation as $yx - xy = 72$, we are making a big mistake.

The correct equation is $(10y + x) - (10x + y) = 72$

i.e. $9y - 9x = 72$ i.e. $y - x = 8$.

Thus the difference between the digits is 8.

E.g. 6: When a two digit number is multiplied with the unit's digit, the product is 144 and when the number is multiplied with the ten's digit, the product is 504. Find the number.

Assuming the number to be *xy*, we get two equation as

$x \times (10x + y) = 504$ and $y \times (10x + y) = 144$.

Dividing the two equations, $\frac{x}{y} = \frac{504}{144} = \frac{7}{2}$

Now one could have substituted $x = \frac{7}{2}y$ in either of the two equations and found the value of x and y. But this is unnecessary waste of time.

Since x and y are digits, they can assume only single digit whole numbers, and the only such values that satisfy $\frac{x}{y} = \frac{7}{2}$ is $x = 7$ and $y = 2$. Thus the number is 72 and one just need to check that 72×7 is indeed 504 and 72×2 is indeed 144.

Whenever variables are assumed to be digits of a number...

What if the above question required us to find the number?

One first sight $y - x = 8$ might appear to have infinite solutions and thus, we may come to the conclusion, there are infinite such numbers. Wrong!

Whenever digits are assumed as variables keep in mind that THE ONLY VALUES THEY CAN TAKE ARE 0, 1, 2, ..., 9. Since they are 'digits', they cannot be decimals and nor can they be 10, 11, ...

Because of this limitation, while there could be infinite solutions to an equation, when the variables used are for digits, the number of solutions gets limited.

E.g. if x and y are any whole numbers, then the equation $y - x = 8$ would have infinite solutions. But if x and y are digits of a two digit number, the only possible solutions to $y - x = 8$ are (8, 0) and (9, 1) for (y, x).

Also keep in mind that when digits of a number are assumed as variables, THE DIGIT IN LEADING POSITION CANNOT BE ZERO. Thus, in the above example $x = 0$ is not possible.

Thus, the original number can be uniquely identified as 19.

This aspect is increasingly being used in entrance exams. See next example for an interesting such limitation.

E.g. 7: The difference between two three-digit numbers is 792. If the two numbers have digits in reverse order, how many such pairs of numbers can be formed?

The two numbers could be taken as *abc* and *cba*. Please keep in mind that neither a nor c can be zero or else the numbers would not be three-digit numbers.

$(100a + 10b + c) - (100c + 10b + a) = 792$ i.e. $99a - 99c = 792 \Rightarrow a - c = 8$

Since a and c are single digit natural numbers, the only value that satisfies the equation is $a = 9$ and $c = 1$.

But b could take any value 0, 1, 2, …, 9 i.e. 10 different values. Thus, in all there could be 10 pairs of numbers.

E.g. 8: How many four-digit numbers of the form *abab*, where a, b, a, b refer to the digits of the number, are perfect squares?

The number can be manipulated as

Given number = $1000a + 100b + 10a + b = 1010a + 101b$

$= 101 \times (10a + b)$ is a perfect square.

This is possible only when $10a + b$ is a multiple of 101.

But the highest value that $10a + b$ can take is 99 since the maximum value that a and b can take is 9. Thus, with a and b being digits, $10a + b$ can never be a multiple of 101 and thus $101 \times (10a + b)$ can never be a perfect square. No number of the form *abab* is a perfect square.

Miscellaneous:

E.g. 9: Raj starts writing all natural numbers starting from 1. Find the 8,000th digit that he writes?

There are a total of 9 single digit numbers (1 to 9) which would mean writing 9 digits.

There are a total of 90 two digit numbers (10 to 99) which would mean writing 180 digits.

There are a total of 900 three digit numbers (100 to 999) which would mean writing 2700 digits.

There are a total of 9000 four digit numbers and so if all are written, we would overshoot writing the 8,000th digit.

Thus the 8,000th digit would be written while writing the four digit numbers.

From 1 to 999, a total of $9 + 180 + 2700 = 2889$ digits would have been written. Further $8000 - 2889 = 5111$ digits have to be written, all of which would be covered while writing four digit numbers. While writing each number from 1000 onwards, four digits would be written for each number. Dividing 5111 by 4 we get the quotient as 1277 and a remainder of 3. Thus Raj will have to write the first 1277 four digit numbers i.e. from 1000 to 2276 and while Raj is writing the number 2277, the 8,000th digit will be 7 (the one in the hundred's place).

E.g. 10: There are some mints kept in a bowl on a table. Amit comes and takes one-third the number of mints in the bowl plus two more mints. Next, Sumit comes and takes one-half the number of mints now in the bowl. But then he feels guilty and returns 2 back to the bowl. Finally Rohit comes along and takes one-fourth of mints now in the bowl and four more. If there are now 8 mints in the bowl, how many mints were there in the bowl before Amit walked in?

When a fraction is taken and focus is on what remains ...

Whenever in a question a fraction is taken away and the question is about the amount left over, one should immediately change the fraction orally ...

"Amit takes one-third" should be read as "two-thirds remain the bowl"

And this can be used in innumerable situations ...

Rajeev spends one fifth of his income on rent. He spends two-ninth of the remaining income on entertainment. If he spends three-seventh of the remaining on education, find what fraction of his income does he save.

After rent, $\frac{4}{5}^{\text{th}}$ of his income remains.

After entertainment, $\frac{7}{9}^{\text{th}}$ of $\frac{4}{5}^{\text{th}}$ of his income remains.

After education, $\frac{4}{7}^{\text{th}}$ of $\frac{7}{9}^{\text{th}}$ of $\frac{4}{5}^{\text{th}}$ of his income remains i.e. Rajeev saves $\frac{16}{45}^{\text{th}}$ of his income.

If, x were the initial number of mints, and the shortcut explained is understood, the equation can be written in one step as $\frac{3}{4}\left[\frac{1}{2}\left(\frac{2}{3}x-2\right)+2\right]-4=8$.

Solving this, one gets $x = 45$.

Shorter Method

One could just proceed backwards ...

8 mints were left after 4 more were taken i.e. before 4 being taken there were 12 mints.

12 mints were left after one-fourth were taken i.e. three-fourth of mints in the bowl is 12, thus mints in the bowl before Rohit came were 16.

16 were after 2 were returned i.e. before returning the mints, there were 14 mints.

14 mints were left after one-half were taken. Thus, mints in the bowl before Sumit came were 28

28 mints were left after 2 mints were taken i.e. before 2 being taken there were 30 mints.

30 mints were left after one-third were taken i.e. two-third of mints in the bowl is 30, thus mints in the bowl before Amit came were 45.

Exercise

1. Find the total number of digits required in numbering the pages of book containing 1024 pages?

   1. 4096 2. 3024 3. 2985 4. 2989 5. None of these

2. A book contains 534 pages. How many times does the digit 3 appear in the page numbers?

   1. 110 2. 108 3. 90 4. 88 5. None of these

3. $\dfrac{n^2 + 2n\sqrt{n} + 8\sqrt{n} + 16}{n + 4\sqrt{n} + 4} = ?$

   1. $n + 2\sqrt{n} + 4$ 2. $n + 2\sqrt{n} - 4$ 3. $n - 2\sqrt{n} - 4$ 4. $n - 2\sqrt{n} + 4$ 5. n

4. The ratio of a two two-digit numbers, which have the same digits but in reverse order, is 7 : 4. What is the maximum possible difference between the two numbers?

   1. 9 2. 18 3. 17 4. 36 5. 45

5. When we multiply a certain two digit number by the sum of its digits, 324 is achieved. If you multiply the number written in reverse order of the same digits by the sum of the digits, we get 567. What is the sum of the digits of the number?

   1. 3 2. 6 3. 9 4. 12 5. 15

6. A bus started from its depot filled to capacity. It stops at point A where one-fifth of the passengers alight and 24 board the bus. At point B, one-fourth of the passengers alight and 12 board the bus. At point C which was the last stop all the 120 passengers alight. Find the capacity of the bus.

   1. 132 2. 144 3. 120 4. 150 5. None of these

7. Let x = 0.123456789101112......998999, where the digits are obtained by writing the integers 1 through 999 in order. Find the 1983rd digit to the right of the decimal point.

   1. 5 2. 6 3. 7 4. 8 5. 9

8. Find the smallest four-digit number which when successively divided by 6, 8 and 10 leave remainders of 4, 5 and 6 respectively.

   1. 1162 2. 1562 3. 1042 4. 1282 5. None of these

Directions for questions 9 & 10: In the ‘alphametic’ each of the letters represents a unique whole number from 0 to 9. No two letters represent the same digit and no digit is represented by two different letters.

9.
$$\begin{array}{rcccccc}
 & & T & H & R & E & E \\
+ & & T & H & R & E & E \\
+ & & & & T & W & O \\
+ & & & & T & W & O \\
+ & & & & O & N & E \\
\hline
 & E & L & E & V & E & N
\end{array}$$

What is the sum E + L + E + V + E + N?

1. 27 2. 24 3. 21 4. 18 5. 15

10.
$$\begin{array}{rccccc}
 & & & T & W & O \\
\times & & & T & W & O \\
\hline
 & T & H & R & E & E
\end{array}$$

What is the sum T + H + R + E + E?

1. 27 2. 24 3. 21 4. 18 5. 15

CAT Questions

1. *[CAT 2008]* A shop stores x kg of rice. The first customer buys half this amount plus half a kg of rice. The second customer buys half the remaining amount plus half a kg of rice. Then the third customer also buys half the remaining amount plus half a kg of rice. Thereafter, no rice is left in the shop. Which of the following best describes the value of x?

   (1) $2 \le x \le 6$ (2) $5 \le x \le 8$ (3) $9 \le x \le 12$ (4) $11 \le x \le 14$ (5) $13 \le x \le 18$

2. *[CAT 2008]* The number of common terms in the two sequences 17, 21, 25, ... , 417 and 16, 21, 26, ... , 466 is

   (1) 78 (2) 19 (3) 20 (4) 77 (5) 22

3. *[CAT 2008]* The integers 1, 2, ..., 40 are written on a blackboard. The following operation is then repeated 39 times: In each repetition, any two numbers, say a and b, currently on the blackboard are erased and a new number $a + b - 1$ is written. What will be the number left on the board at the end?

   (1) 820 (2) 821 (3) 781 (4) 819 (5) 780

4. *[CAT 2008]* Suppose, the seed of any positive integer n is defined as follows:

   seed(n) = n, if $n < 10$

   = seed($s(n)$), otherwise, where $s(n)$ indicates the sum of digits of n.

   For example, seed(7) = 7, seed(248) = seed(2 + 4 + 8) = seed(14) = seed(1 + 4) = seed(5) = 5 etc.

   How many positive integers n, such that $n < 500$, will have seed(n) = 9?

   (1) 39 (2) 72 (3) 81 (4) 108 (5) 55

5. *[CAT 2008]* What are the last two digits of 7^{2008}?

   (1) 21 (2) 61 (3)01 (4)41 (5)81

6. *[CAT 2008]* Three consecutive positive integers are raised to the first, second and third powers respectively and then added. The sum so obtained is a perfect square whose square root equals the total of the three original integers. Which of the following best describes the minimum, say m, of these three integers?

   (1) $1 \le m \le 3$ (2) $4 \le m \le 6$ (3) $7 \le m \le 9$ (4) $10 \le m \le 12$ (5) $13 \le m \le 15$

7. *[CAT 2007]* Consider four digit numbers for which the first two digits are equal and the last two digits are also equal. How many such numbers are perfect squares?

   (1) 2 (2) 4 (3) 0 (4) 1 (5) 3

8. *[CAT 2007]* How many pairs of positive integers m, n satisfy $\frac{1}{m} + \frac{4}{n} = \frac{1}{12}$, where n is an odd integer less than 60?

   (1) 4 (2) 7 (3) 5 (4) 3 (5) 6

9. *[CAT 2006]* If $x = -0.5$, then which of the following has the smallest value?

   (1) $2^{\frac{1}{x}}$ (2) $\frac{1}{x}$ (3) $\frac{1}{x^2}$ (4) 2^x (5) $\frac{1}{\sqrt{-x}}$

10. *[CAT 2006]* Which among $2^{1/2}$, $3^{1/3}$, $4^{1/4}$, $6^{1/6}$ and $12^{1/12}$ is the largest?

(1) $2^{1/2}$ (2) $3^{1/3}$ (3) $4^{1/4}$ (4) $6^{1/6}$ (5) $12^{1/12}$

11. *[CAT 2006]* The sum of four consecutive two digit odd numbers, when divided by 10, becomes a perfect square. Which of the following can possibly be one of these four numbers?

(1) 21 (2) 25 (3) 41 (4) 67 (5) 73

12. *[CAT 2006]* When you reverse the digits of the number 13, the number increases by 18. How many other two digit numbers increase by 18 when their digits are reversed?

(1) 5 (2) 6 (3) 7 (4) 8 (5) 10

13. *[CAT 2006]* The number of employees in Obelix Menhir Co. is a prime number and is less than 300. The ratio of the number of employees who are graduates and above, to that of employees who are not, can possibly be:

(1) 101 : 88 (2) 87 : 100 (3) 110 : 111 (4) 85 : 98 (5) 97 : 84

14. *[CAT 2005]* If $x = (16^3 + 17^3 + 18^3 + 19^3)$, then x divided by 70 leaves a remainder of

(1) 0 (2) 1 (3) 69 (4) 35

15. *[CAT 2005]* If $R = \dfrac{30^{65} - 29^{65}}{30^{64} + 29^{64}}$, then

(1) $0 < R \le 0.1$ (2) $0.1 < R \le 0.5$ (3) $0.5 < R \le 1.0$ (4) $R > 1.0$

16. *[CAT 2005]* Let $n! = 1 \times 2 \times 3 \times \ldots \times n$ for integer $n \ge 1$. If $p = 1! + (2 \times 2!) + (3 \times 3!) + \ldots\ldots + (10 \times 10!)$, then $p + 2$ when divided by 11! leaves a remainder of

(1) 10 (2)0 (3)7 (4)1

17. *[CAT 2005]* The digits of a three-digit number A are written in the reverse order to form another three-digit number B. If $B > A$ and $B - A$ is perfectly divisible by 7, then which of the following is necessarily true?

(1) $100 < A < 299$ (2) $106 < A < 305$ (3) $112 < A < 311$ (4) $118 < A < 317$

18. *[CAT 2005]* The rightmost non-zero digit of the number 30^{2720} is

(1) 1 (2) 3 (3) 7 (4) 9

19. *[CAT 2005]* For a positive integer n, let p_n denote the product of the digits of n, and s_n denote the sum of the digits of n. The number of integers between 10 and 1000 for which $p_n + s_n = n$ is

(1) 81 (2) 16 (3) 18 (4) 9

20. *[CAT 2004]* The total number of integer pairs (x, y) satisfying the equation $x + y = xy$ is

(1) 0 (2) 1 (3) 2 (4) None of the above

21. *[CAT 2004]* The remainder, when $(15^{23} + 23^{23})$ is divided by 19, is

(1) 4 (2) 15 (3) 0 (4) 18

22. *[CAT 2003L]* How many even integers n, where $100 \le n \le 200$, are divisible neither by seven nor by nine?

(1) 40 (2) 37 (3) 39 (4) 38

23. *[CAT 2003L]* Answer the question on the basis of the information given below.

Choose 1 if the question can be answered by one of the statements alone but not by the other.
Choose 2 if the question can be answered by using either statement alone.
Choose 3 if the question can be answered by using both the statements together, but cannot be answered by using either statement alone.
Choose 4 if the question cannot be answered even by using both the statements together.

Is $a^{44} < b^{11}$, given that $a = 2$ and b is an integer?

Statement A: b is even   Statement B: b is greater than 16

24. *[CAT 2003L]* The number of positive integers n in the range $12 \le n \le 40$ such that the product

$(n-1)(n-2) \ldots 3.2.1$ is not divisible by n is _____.

(1) 5   (2) 7   (3) 13   (4) 14

25. *[CAT 2003L]* Let T be the set of integers {3, 11, 19, 27, ..., 451, 459, 467} and S be a subset of T such that the sum of no two elements of S is 470. The maximum possible number of elements in S is

(1) 32   (2) 28   (3) 29   (4) 30

26. *[CAT 2003R]* Consider the sets $T_n = \{n, n+1, n+2, n+3, n+4)$, where $n = 1, 2, 3, \ldots\ldots, 96$. How many of these sets contain 6 or any integral multiple thereof (i.e. any one of the numbers 6, 12, 18, ...)?

(1) 80   (2) 81   (3) 82   (4) 83

27. *[CAT 2003R]* A real number x satisfying $1-\frac{1}{n} < x \le 3+\frac{1}{n}$, for every positive integer n, is best described by:

(1) $1 < x < 4$ (2) $1 < x \le 3$ (3) $0 < x \le 4$ (4) $1 \le x \le 3$

28. *[CAT 2003R]* What is the remainder when 4^{96} is divided by 6?

(1) 0   (2) 2   (3) 3   (4) 4

DIRECTIONS for Questions 29 to 31: *[CAT 2003R]* Answer the questions on the basis of the information given below.

The seven basic symbols in a certain numeral system and their respective values are as follows:
I = 1, V = 5, X = 10, L = 50, C = 100, D = 500, and M = 1 000

In general, the symbols in the numeral system are read from left to right, starting with the symbol representing the largest value; the same symbol cannot occur contiguously more than three times; the value of the numeral is the sum of the values of the symbols. For example, XXVII = 10 + 10 + 5 + 1 + 1 = 27. An exception to the left-to-right reading occurs when a symbol is followed immediately by a symbol of greater value; then, the smaller value is subtracted from the larger. For example. XLVI = (50 – 10) + 5 + 1 = 46.

29. The value of the numeral MDCCLXXXVII is:

(1) 1687   (2) 1787   (3) 1887   (4) 1987

30. The value of the numeral MCMXCIX is:

(1) 1999   (2) 1899   (3) 1989   (4) 1889

31. Which of the following can represent the numeral for 1995?

a. MCMLXXV   b. MCMXCV   c. MVD   d. MVM

(1) only (a) & (b)   (2) only (c) & (d)   (3) only (b) & (d)   (4) only (d)

32. *[CAT 2003R]* What is the sum of all two-digit numbers that give a remainder of 3 when they are divided by 7?

(1) 666 (2) 676 (3) 683 (4) 777

33. *[CAT 2003R]* Let x and y be positive integers such that x is prime and y is composite. Then,

(1) $y - x$ cannot be an even integer. (2) xy cannot be an even integer.

(3) $(x + y)/x$ cannot be an even integer. (4) None of the above statements is true.

34. *[CAT 2003R]* Let n (>1) be a composite integer such that $\sqrt{n}$ is not an integer. Consider the following statements:

A: n has a perfect integer-valued divisor which is greater than 1 and less than $\sqrt{n}$

B: n has a perfect integer-valued divisor which is greater than $\sqrt{n}$ but less than n

Then,

(1) Both A and B are false. (2) A is true but B is false.

(3) A is false but B is true. (4) Both A and B are true.

35. *[CAT 2003R]* If a, $a + 2$, and $a + 4$ are prime numbers, then the number of possible solutions for a is:

(1) one (2) two (3) three (4) more than three

36. *[CAT 2002]* If $pqr = 1$, the value of the expression $\frac{1}{1+p+q^{-1}}+\frac{1}{1+q+r^{-1}}+\frac{1}{1+r+p^{-1}}$ is equal to

(1) $p + q + r$ (2) $1/(p + q + r)$ (3) 1 (4) $p^{-1} + q^{-1} + r^{-1}$

37. *[CAT 2002]* $7^{6n} - 6^{6n}$, where n is an integer > 0, is divisible by

(1) 13 (2) 127 (3) 559 (4) All of these

38. *[CAT 2002]* If u, v, w and m are natural numbers such that $u^m + v^m = w^m$, then one of the following is true.

(1) $m > \min(u, v, w)$ (2) $m > \max(u, v, w)$ (3) $m < \min(u, v, w)$ (4) none of these

39. *[CAT 2002]* After the division of a number successively by 3, 4 and 7, the remainders obtained are 2, 1 and 4 respectively. What will be the remainder if 84 divides the same number?

(1) 80 (2) 76 (3) 41 (4) 53

40. *[CAT 2002]* Three pieces of cakes of weight $4\frac{1}{2}$ lbs, $6\frac{3}{4}$ lbs and $7\frac{1}{5}$ lbs respectively are to be divided into parts of equal weights. Further, each part must be as heavy as possible. If one such part is served to each guest, then what is the maximum number of guests that could be entertained?

(1) 54 (2) 72 (3) 20 (4) none of these

41. *[CAT 2002]* At a bookstore, "MODERN BOOK STORE" is flashed using neon lights. The words are individually flashed at intervals of $2\frac{1}{2}$, $4\frac{1}{4}$, $5\frac{1}{8}$ seconds respectively and each word is put off after a second. The least time after which the full name of the bookstore can be read again is:

(1) 49.5 seconds (2) 73.5 seconds (3) 1744.5 seconds (4) 855 seconds

42. *[CAT 2002]* When 2^{256} is divided by 17 the remainder would be

(1) 1 (2) 16 (3) 14 (4) none of these

43. *[CAT 2002]* Amol was asked to calculate the arithmetic mean of ten positive integers each of which had two digits. By mistake, he interchanged the two digits, say a and b, in one of these ten integers. As a result, his answer for the arithmetic mean was 1.8 more than what it should have been. Then $b - a$ equals

    (1) 1 (2) 2 (3) 3 (4) none of these

44. *[CAT 2002]* The owner of a local jewellery store hired 3 watchmen to guard his diamonds, but a thief still got in and stole some diamonds. On the way out, the thief met each watchman, one at a time. To each he gave ½ of the diamonds he had then, and 2 more besides. He escaped with one diamond. How many did he steal originally?

    (1) 40 (2) 36 (3) 25 (4) none of these

45. *[CAT 2002]* Number S is obtained by squaring the sum of digits of a two digit number D. If difference between S and D is 27, then the two digit number D is:

    (1) 24 (2) 54 (3) 34 (4) 45

46. *[CAT 2001]* Let x, y and z be distinct integers, x and y are odd and positive, and z is even and positive. Which one of the following statements cannot be true?

    (1) $(x - z)^2 y$ is even (2) $(x - z)y^2$ is odd (3) $(x - z)y$ is odd (4) $(x - y)^2 z$ is even

47. *[CAT 2001]* A red light flashes 3 times per minute and a green light flashes 5 times in two minutes at regular intervals. If both lights start flashing at the same time, how many times do they flash together in each hour?

    (1) 30 (2) 24 (3) 20 (4) 60

48. *[CAT 2001]* In a 4-digit number, the sum of the first two digits is equal to that of the last two digits. The sum of the first and last digits is equal to the third digit. Finally, the sum of the second and fourth digits is twice the sum of the other two digits. What is the third digit of the number?

    (1) 5 (2) 8 (3) 1 (4) 4

49. *[CAT 2001]* Anita had to do a multiplication. Instead of taking 35 as one of the multipliers, she took 53. As a result, the product went up by 540. What is the new product?

    (1) 1050 (2) 540 (3) 1440 (4) 1590

50. *[CAT 2001]* Three friends, returning from a movie, stopped to eat at a restaurant. After inner, they paid their bill and noticed a bowl of mints at the front counter. Sita took 1/3 of the mints, but returned four because she had a momentary pang of guilt. Fatima then took 1/4 of what was left but returned three for similar reasons. Eswari then took half of the remainder but threw two back into the bowl. The bowl had only 17 mints left when the raid was over. How many mints were originally in the bowl?

    (1) 38 (2) 31 (3) 41 (4) None of these

51. *[CAT 2001]* Let b be a positive integer and $a = b^2 - b$. If $b > 4$, then $a^2 - 2a$ is divisible by

    (1) 15 (2) 20 (3) 24 (4) None of these

52. *[CAT 2000]* Let D be a recurring decimal of the form, D = $0.a_1a_2a_1a_2a_1a_2$...... , where digits a_1 and a_2 lie between 0 and 9. Further, at most one of them is zero. Then which of the following numbers necessarily produces an integer, when multiplied by D?

    (1) 18 (2) 108 (3) 198 (4) 288

53. *[CAT 2000]* Let S be the set of integers x such that
(i) $100 < x < 200$ (ii) x is odd (iii) x is divisible by 3 but not by 7
How many elements does S contain?
(1) 16 (2) 12 (3) 11 (4) 13

54. *[CAT 2000]* Let x, y and z be distinct integers, that are odd and positive. Which one of the following statements cannot be true?
(1) xyz^2 is odd (2) $(x - y)^2z$ is even
(3) $(x + y - z)^2(x + y)$ is even (4) $(x - y)(y + z)(x + y - z)$ is odd

55. *[CAT 2000]* Let S be the set of prime numbers greater than or equal to 2 and less than 100. Multiply all elements of S. With how many consecutive zeros will the product end?
(1) 1 (2) 4 (3) 5 (4) 10

56. *[CAT 2000]* Let N = 1421 × 1423 × 1425. What is the remainder when N is divided by 12?
(1) 0 (2) 9 (3) 3 (4) 6

57. *[CAT 2000]* The integers 34041 and 32506 when divided by a three-digit integer n leave the same remainder. What is n?
(1) 289 (2) 367 (3) 453 (4) 307

58. *[CAT 2000]* Each of the numbers $x_1, x_2, \ldots\ldots, x_n$ where $n > 4$, is equal to 1 or –1. Suppose,
$x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + \ldots\ldots + x_{n-3}x_{n-2}x_{n-1}x_n + x_{n-2}x_{n-1}x_nx_1 + x_{n-1}x_nx_1x_2 + x_nx_1x_2x_3 = 0$, then,
(1) n is even (2) n is odd
(3) n is an odd multiple of 3 (4) n is prime

59. *[CAT 2000]* Let $N = 55^3 + 17^3 - 72^3$. N is divisible by
(1) both 7 and 13 (2) both 3 and 13 (3) both 17 and 7 (4) both 3 and 17

60. *[CAT 1999]* Let a, b, c be distinct digits. Consider a two digit number '*ab*' and a three digit number '*ccb*' both defined under the usual decimal number system. If $(ab)^2 = ccb$ and $ccb > 300$ then the value of b is
(1) 1 (2) 0 (3) 5 (4) 6

61. *[CAT 1999]* The remainder when 7^{84} is divided by 342 is
(1) 0 (2) 1 (3) 49 (4) 341

62. *[CAT 1999]* If $n = 1 + x$, where x is the product of four consecutive positive integers, then which of the following is/are true?
A. n is odd B. n is prime C. n is a perfect square
(1) A and C only (2) A and B only (3) A only (4) None of the above.

63. *[CAT 1999]* For two positive integers a and b define the function $h(a, b)$ as the greatest common factor (gcf) of a, b. Let A be a set of n positive integers. G(A), the gcf of the elements of set A is computed by repeatedly using the function h. The minimum number of times h is required to be used to compute G is
(1) $\frac{1}{2}n$ (2) $(n - 1)$ (3) n (4) None of the above

64. *[CAT 1999]* If $n^2 = 123456787654321$, what is n?
(1) 12344321 (2) 1235789 (3) 11111111 (4) 1111111

Directions for questions 65 to 67: *[CAT 1999]* These questions are based on the situation given below

A young girl Roopa leaves home with x flowers, goes to the bank of a nearby river. On the bank of the river, there are four places of worship, standing in a row. She dips all the x flowers into the river. The number of flowers doubles. Then she enters the first place of worship, offers y flowers to the deity. She dips the remaining flowers into the river, and again the number of flowers doubles. She goes to the second place of worship, offers y flowers to the deity. She dips the remaining flowers into the river, and again the number of flowers doubles.

She goes to the third place of worship, offers y flowers to the deity. She dips the remaining flowers into the river, and again the number of flowers doubles. She goes to the fourth place of worship, offers y flowers to the deity. Now she is left with no flower in hand.

65. If Roopa leaves home with 30 flowers, the number of flowers she offers to each deity is

    (1) 30 (2) 31 (3) 32 (4) 33

66. The minimum number of flowers that could be offered to each deity is

    (1) 0 (2) 15 (3) 16 (4) Cannot be determined

67. The minimum number of flowers with which Roopa leaves home is

    (1) 16 (2) 15 (3) 0 (4) Cannot be determined

Directions for questions 68 to 70: *[CAT 1999]* These questions are based on the situation given below.

There are fifty integers a_1, a_2, ..., a_{50}, not all of them necessarily different. Let the greatest integer of these fifty integers be referred to as G and smallest integer be referred to as L. The integers a_1, through a_{24} form sequence S_1, and the rest form sequence S_2. Each member of S_1 is less than or equal to each member of S_2.

68. All value in S_1 are changed in sign, while those in S_2 remain unchanged. Which of the following statements is true?

    (1) Every member of S_1 is greater than or equal to every member of S_2.

    (2) G is in S_1

    (3) If all numbers originally in S_1 and S_2 had the same sign, then after the change of sign, the largest number of S_1 and S_2 is in S_1.

    (4) None of the above.

69. Elements of S_1 are in ascending order, and those of S_2 are in descending order, a_{24} and a_{25} are interchanged. Then which of the following statements is true?

    (1) S_1 continues to be in ascending order.

    (2) S_2 continues to be in descending order.

    (3) S_1 continues to be in ascending order and S_2 in descending order.

    (4) None of these.

70. Every element of S_1 is made greater than or equal to every element of S_2 by adding to each element of S_1 an integer x. Then x cannot be less than

    (1) 2^{10} (2) The smallest value of S_2

    (3) The largest value of S_2 (4) (G – L)

Answer Key

Classification

1. i. 123/999 ii. 122/990 iii. 111/900
iv. 101/990 v. 92/900 vi. 102/9990
vii. 102/9999 viii. 101/9900
ix. 120/9999 x. 2/9900

2. 2 3. 3 4. 4 5. 1 6. 1 7. 2 8. 2
9. 1 10. 3 11. 1 12. 2 13. 3 14. 2 15. 1 16. 1
17. 4 18. F, F, F, F 19.i. 3 19.ii. 1
20. 4 21. 1 22. 2 23. 3 24. 2 25. 2

Divisibility Rules

1. 3 2. 1 3. 4 4. 4 5. 4 6. 3 7. 4 8. 4
9. 1 10. 3

Indices

1. 1 2. 4 3. 3 4. 3 5. 3 6. 4 7. 4 8. 3
9. 3 10. 3 11. 4 12. 2

Cyclicity

1. i. 7 ii. 7 iii. 9 2. 1 3. 5 4. 1 5. 5 6. 3
7. 2 8. 3 9. 1 10. 2

Factorials

1. 2 2. 4 3. 3 4. 3 5. 2 6. 2 7. 3 8. 4
9. 1 10. 4 11. 5 12. 4 13. 4 14. 4 15. 1 16. 5
17. 5 18. 4 19. 3 20. 1 21. 4 22. 2 23. 4 24. 4
25. 4

Factorisation

1. 5 2. 4 3. 3 4. 1 5. 1 6. 4 7. 4 8. 3

Number of Factors

1. 4 2. 1 3. 3 4. 4 5. 5 6. 2 7. 4 8. 4
9. 3 10. 5 11. 4 12. 2 13. 4 14. 5 15. 4 16. 5
17. 3 18. 1 19. 2 20. 3 21. 4 22. 5 23. 4 24. 1
25. 1 26. 2 27. 1 28. 1 29. 2 30. 2 31. 3 32. 4
33. 5

HCF & LCM

1. 5 2. 4 3. 2 4. 2 5. 2 6. 2 7. 4 8. 3
9. 5 10. 5 11. 4 12. 4 13. 3 14. 5 15. 3 16. 1
17. 3 18. 2 19. 5 20. 3

Remainders

1. 5 2. 1 3. 1 4. 4 5. 4 6. 5 7. 5 8. 4
9. 2 10. 1 11. 4 12. 3 13. 4 14. 3 15. 1 16. 1
17. 4 18. 3 19. 4 20. 4 21. 1 22. 4 23. 3 24. 4
25. 4 26. 3

Miscellaneous

1. 4 2. 5 3. 4 4. 4 5. 3 6. 4 7. 3 8. 4
9. 3 10. 4

CAT Questions

1. 2 2. 3 3. 3 4. 5 5. 3 6. 1 7. 4 8. 4
9. 2 10. 2 11. 3 12. 2 13. 5 14. 1 15. 4 16. 4
17. 2 18. 1 19. 4 20. 3 21. 3 22. 3 23. 1 24. 2
25. 4 26. 1 27. 4 28. 4 29. 2 30. 1 31. 3 32. 2
33. 4 34. 4 35. 1 36. 3 37. 4 38. 4 39. 4 40. 4
41. 2 42. 1 43. 2 44. 2 45. 2 46. 1 47. 1 48. 1
49. 4 50. 4 51. 3 52. 3 53. 4 54. 4 55. 1 56.3
57. 4 58. 1 59. 4 60. 1 61. 2 62. 1 63. 2 64. 3
65. 3 66. 3 67. 2 68. 4 69. 1 70. 4