14

Inequalities

This chapter will seem to be highly mathematical to you when you read the theory contained in the chapter and look at the solved examples. For students weak in Math, there is no need to be disheartened about the seemingly high mathematical content. I would advise you to go through this chapter and internalise the concepts. However, keep in mind the fact that in an aptitude test, the questions will have options, and with options all you will need to do will be check the validity of the inequality for the different options.

In fact, the questions in this chapter have options on both the levels and with option-based solutions, all these questions will seem easy to you.

However, I would advise students aiming to score high marks in Quantitative Ability to try to mathematically solve all the questions on all three levels in this chapter (even though option-based solution will be much easier.)

Two real numbers or two algebraic expressions related by the symbol > ("Greater Than") or < ("Less Than") (and also by the signs ≥ or ≤) form an inequality.

$A < B$, $A > B$ (are plain inequalities)

$A \geq B$, $A \leq B$ (are called as inequations)

The inequality consists of two sides—the left hand side, A and the right hand side, B. A and B can be algebraic expressions or they can be numbers.

An inequality with the < or > sign is called a *strict inequality* while an inequality having ≥ or ≤ sign is called a *slack inequality*. The expressions A and B have to be considered on the set where A and B have sense simultaneously. This set is called the set of permissible values of the inequality. If the terms on the LHS and the RHS are algebraic equations/identities, then the inequality may or may not hold true for a particular value of the variable/set of variables assumed.

The direction in which the inequality sign points is called *the sense of the inequality*. If two or several inequalities contain the same sign (< or >) then they are called *inequalities of the same sense*. Otherwise they are called *inequalities of the opposite sense*.

Now let us consider some basic *definitions* about inequalities.

For 2 real numbers a and b

The inequality $a > b$ means that the difference $a - b$ is positive.

The inequality $a < b$ means that the difference $a - b$ is negative.

PROPERTIES OF INEQUALITIES

For any two real numbers a and b, only one of the following restrictions can hold true:

$$a = b,\ a > b \text{ or } a < b$$

Definitions of Slack Inequalities

The inequality $a \geq b$ means that $a > b$ or $a = b$, that is, a is not less than b.

The inequality $a \leq b$ means that $a < b$ or $a = b$, that is, a is not greater than b.

We can also have the following double inequalities for simultaneous situations:

$$a < b < c,\ a < b \leq c,\ a \leq b < c,\ a \leq b \leq c$$

Properties of Inequalities

1. If $a > b$ then $b < a$ and vice versa.
2. If $a > b$ and $b > c$ then $a > c$.
3. If $a > b$ then for any c, $a + c > b + c$. In other words, an inequality remains true if the same number is added on both sides of the inequality.

Contd

Properties of Inequalities (*Contd*)

4. Any number can be transposed from one side of an inequality to the other side of the inequality with the sign of the number reversed. This does not change the sense of the inequality.
5. If $a > b$ and $c > 0$ then $ac > bc$. Both sides of an inequality may be multiplied (or divided) by the same positive number without changing the sense of the inequality.
6. If $a > b$ and $c < 0$ then $ac < bc$. That is, both sides of an inequality may be multiplied (or divided) by the same negative number but then the sense of the inequality is reversed.
7. If $a > b$ and $c > d$ then $a + c > b + d$. (Two inequalities having the same sense may be added termwise.)
8. If $a > b$ and $c < d$ then $a - c > b - d$
   From one inequality it is possible to subtract termwise another inequality of the opposite sense, retaining the sense of the inequality from which the other was subtracted.
9. If a, b, c, d are positive numbers such that $a > b$ and $c > d$ then $ac > bd$, that is, two inequalities of the same sense in which both sides are positive can be multiplied termwise, the resulting inequality having the same sense as the multiplied inequalities.
10. If a and b are positive numbers where $a > b$, then $a^n > b^n$ for any natural n.
11. If a and b are positive numbers where $a > b$ then $a^{1/n} > b^{1/n}$ for any natural $n \geq 2$.
12. Two inequalities are said to be equivalent if the correctness of one of them implies the correctness of the other, and vice versa.
    Students are advised to check these properties with values and form their own understanding and language of these rules.

Certain Important Inequalities

1. $a^2 + b^2 \geq 2ab$ (Equality for $a = b$)
2. $|a + b| \leq |a| + |b|$ (Equality reached if both a and b are of the same sign or if one of them is zero.)
   This can be generalised as $|a_1 + a_2 + a_3 + \ldots + a_n| \leq |a_1| + |a_2| + |a_3| + \ldots\ldots|a_n|$
3. $|a - b| \geq |a| - |b|$
4. $ax^2 + bx + c \geq 0$ if $a > 0$ and $D = b^2 - 4ac \leq 0$. The equality is achieved only if $D = 0$ and $x = -b/2a$.
5. Arithmetic mean $\geq$ Geometric mean. That is,
   $$\frac{(a+b)}{2} \geq ab$$
6. $a/b + b/a \geq 2$ if $a > 0$ and $b > 0$ or if $a < 0$ and $b < 0$, that is, both a and b have the same sign.

Contd

Certain Important Inequalities (*Contd*)

7. $a^3 + b^3 \geq ab(a + b)$ if $a > 0$ and $b > 0$, the equality being obtained only when $a = b$.
8. $a^2 + b^2 + c^2 \geq ab + ac + bc$
9. $(a + b)(b + c)(a + c) \geq 8\,abc$ if $a \geq 0$, $b \geq 0$ and $c \geq 0$, the equation being obtained when $a = b = c$
10. For any 4 numbers x_1, x_2, y_1, y_2 satisfying the conditions
    $$x_1^2 + x_2^2 = 1$$
    $$y_1^2 + y_2^2 = 1$$
    the inequality $|x_1y_1 + x_2y_2| \leq 1$ is true.
11. $\dfrac{a}{b^{1/2}} + \dfrac{b}{a^{1/2}} \geq a^{1/2} + b^{1/2}$ where $a \geq 0$ and $b \geq 0$
12. If $a + b = 2$, then $a^4 + b^4 \geq 2$
13. The inequality $|x| \leq a$, means that
    $$-a \leq x \leq a \text{ for } a > 0$$
14. $2^n > n^2$ for $n \geq 5$

Some Important Results

- If $a > b$, then it is evident that
  $$a + c > b + c$$
  $$a - c > b - c$$
  $$ac > bc$$
  $a/c > b/c$; that is,
  an inequality will still hold after each side has been increased, diminished, multiplied, or divided by the same positive quantity.
  If $a - c > b$,
  By adding c to each side,
  $a > b + c$; which shows that
  in an inequality any term may be transposed from one side to the other if its sign is changed.
- If $a > b$, then evidently $b < a$; that is,
  if the sides of an inequality be transposed, the sign of inequality must be reversed.
- If $a > b$, then $a - b$ is positive, and $b - a$ is negative; that is, $-a-(-b)$ is negative, and therefore $-a < -b$; hence,
  if the signs of all the terms of an inequality be changed, the sign of inequality must be reversed.
  Again, if $a > b$, then $-a < -b$ and, therefore, $-ac < -bc$; that is,
  if the sides of an inequality be multiplied by the same negative quantity, the sign of inequality must be reversed.

Contd

Some Important Results (*Contd*)

If $a_1 > b_1, a_2 > b_2, a_3 > b_3, \ldots a_m > b_m$, it is clear that $a_1 + a_2 + a_3 + \ldots + a_m > b_1 + b_2 + b_3 + \ldots + b_m$; and $a_1\, a_2\, a_3 \ldots a_m > b_1 b_2 b_3 \ldots b_m$.

- If $a > b$, and if p, q are positive integers, then or $a^{1/q} > b^{1/q}$ and, therefore, $a^{p/q} > b^{p/q}$; that is, $a^n > b^n$, where n is any positive quantity. Further,

$$1/a^n < 1/b^n; \text{ that is } a^{-n} < b^{-n}$$

The square of every real quantity is positive, and therefore greater than zero. Thus $(a - b)^2$ *is positive.*

Let a and b be two positive quantities, S their sum and P their product. Then from the identity

$$4ab = (a + b)^2 - (a - b)^2$$

we have $4P = S^2 - (a - b)^2$, and $S^2 = 4P + (a - b)^2$

Hence, if S is given, P is greatest when $a = b$; and if P is given, S is least when $a = b$;
That is, *if the sum of two positive quantities is given, their product is greatest when they are equal; and if the product of two positive quantities is given, their sum is least when they are equal.*

To Find the Greatest Value of a Product, the Sum of Whose Factors is Constant

Let there be n factors $a, b, c, \ldots n$, of a composite number and suppose that their sum is constant and equal to S.

Consider the product $abc\ldots.n$, and suppose that a and b are any two unequal factors. If we replace the two unequal factors a and b by the two equal factors $(a + b)/2$, and $(a + b)/2$, the product is increased while the sum remains unaltered. Hence, so *long as the product contains two unequal factors it can be increased altering the sum of the factors; therefore, the product is greatest when all the factors are equal.* In this case the value of each of the n factors is S/n, and the greatest value of the product is $(S/n)^n$, or $\{(a + b + c + \ldots.. + n/n\}^n$

This will be clearer through an example.

Let us define a number as $a \times b = c$ such that we restrict $a + b = 100$ (maximum).

Then, the maximum value of the product will be achieved if we take the value of a and b as 50 each.

Thus $50 \times 50 = 2500$ will be the highest number achieved for the restriction $a + b \le 100$.

Further, you can also say that $50 \times 50 > 51 \times 49 > 52 \times 48 > 53 \times 47 > 54 \times 46 > \ldots > 98 \times 2 > 99 \times 1$

Thus if we have a larger multiplication as

$4 \times 6 \times 7 \times 8$ this will always be less than $5 \times 5 \times 7 \times 8$. [Holds true only for positive numbers.]

Corollary If $a, b, c\ldots k$, are unequal, $\{(a + b + c + \ldots + k)/n\}^n > abc \ldots k$;
that is , $(a + b + c + \ldots + k)/n > (abc\ldots k)^{1/n}$.

By an extension of the meaning of the arithmetic and geometric means this result is usually quoted as follows: *The arithmetic mean of any number of positive quantities is greater than the geometric mean.*

Definition of Solution of an Inequality

The solution of an inequality is the value of an unknown for which this inequality reduces to a true numerical identity. That is, to solve an inequality means to find all the values of the variable for which the given inequality is true.

An inequality has no solution if there is no such value for which the given inequality is true.

Equivalent Inequalities: Two inequalities are said to be equivalent if any solution of one is also a solution of the other and vice versa.

If both inequalities have no solution, then they are also regarded to be equivalent.

To solve an inequality we use the basic properties of an inequality which have been illustrated above.

Notation of Ranges

1. Ranges Where the Ends are Excluded If the value of x is denoted as (1, 2) it means $1 < x < 2$ i.e. x is greater than 1 but smaller than 2.

Similarly, if we denote the range of values of x as $-(7, -2)U(3, 21)$, this means that the value of x can be denoted as $-7 < x < -2$ and $3 < x < 21$. This would mean that the inequality is satisfied between the two ranges and is not satisfied outside these two ranges.

Based on this notation write the ranges of *x* for the following representations:

$$(1, +\infty)\ U\ (-\infty, -7)$$
$$(-\infty, 0)\ U\ (4, +\infty),\ (-\infty, 50)\ U\ (-50, +\infty)$$

2. Ranges where the Ends are Included

$$[2, 5] \text{ means } 2 \le x \le 5$$

3. Mixed Ranges

$$(3, 21] \text{ means } 3 < x \le 21$$

Solving Linear Inequalities in one Unknown

A linear inequality is defined as an inequality of the form

$$ax + b\ I\ 0$$

where the symbol 'I' represents any of the inequalities $<, >, \geq, \leq$.

For instance if $ax + b \leq 0$, then $ax \leq -b$
$\rightarrow x \leq -b/a$ if $a > 0$ and $x \geq -b/a$ if $a < 0$

Example: Solve the inequality $2(x - 3) - 1 > 3(x - 2) - 4(x + 1)$

$\rightarrow 2x - 7 > 3x - 6 - 4x - 4 \rightarrow 3x > -3$. Hence, $x > -1$

This can be represented in mathematical terms as $(-1, +\infty)$

Example: Solve the inequality $2(x - 1) + 1 > 3 - (1 - 2x)$ $\rightarrow 2x - 1 > 2 + 2x \rightarrow 0.x > 3 \rightarrow$ This can never happen. Hence, no solution.

Example: Solve the inequality $2(x - 1) + 1 < 3 - (1 - 2x)$

Gives: $0.x < 3$.

This is true for all values of x

Example: Solve the inequality $ax > a$.

This inequality has the parametre a that needs to be investigated further.

If $a > 0$, then $x > 1$
If $a < 0$, then $x < 1$

Solving Quadratic Inequalities

A quadratic inequality is defined as an inequality of the form:

$$ax^2 + bx + c\ I\ 0\ (a \neq 0)$$

where the symbol I represents any of the inequalities $<, >, \geq, \leq$.

For a quadratic expression of the form $ax^2 + bx + c$, $(b^2 - 4ac)$ is defined as the discriminant of the expression and is often denoted as D. i.e. $D = b^2 - 4ac$

The following cases are possible for the value of the quadratic expression:

***Case 1:* If $D < 0$**

1. If $a < 0$ then $ax^2 + bx + c < 0$ for all x
2. If $a > 0$ then $ax^2 + bx + c > 0$ for all x.

In other words, we can say that if D is negative then the values of the quadratic expression takes the same sign as the coefficient of x^2.

This can also be said as

If $D < 0$ then all real values of x are solutions of the inequalities $ax^2 + bx + c > 0$ and $ax^2 + bx + c \geq 0$ for $a > 0$ and have no solution in case $a < 0$.

Also, for $D < 0$, all real values of x are solutions of the inequalities $ax^2 + bx + c < 0$ and $ax^2 + bx + c \leq 0$ if $a < 0$ and these inequalities will not give any solution for $a > 0$.

***Case 2:* $D = 0$**

If the discriminant of a quadratic expression is equal to zero, then the value of the quadratic expression takes the same sign as that of the coefficient of x^2 (except when $x = -b/2a$ at which point the value of the quadratic expression becomes 0).

We can also say the following for $D = 0$:

1. The inequality $ax^2 + bx + c > 0$ has as a solution any $x \neq -(b/2a)$ if $a > 0$ and has no solution if $a < 0$.
2. The inequality $ax^2 + bx + c < 0$ has as a solution any $x \neq -(b/2a)$ if $a < 0$ and has no solution if $a > 0$.
3. The inequality $ax^2 + bx + c \geq 0$ has as a solution any x if $a > 0$ and has a unique solution $x = -b/2a$ if $a < 0$.
4. The inequality $ax^2 + bx + c \leq 0$ has as a solution any x if $a < 0$ and has a unique solution $x = -b/2a$ for $a > 0$.

***Case 3:* $D > 0$**

If x_1 and x_2 are the roots of the quadratic expression then it can be said that:

1. For $a > 0$, $ax^2 + bx + c$ is positive for all values of x outside the interval $[x_1, x_2]$ and is negative for all values of x within the interval (x_1, x_2). Besides for values of $x = x_1$ or $x = x_2$, the value of the quadratic expression becomes zero (By definition of the root).
2. For $a < 0$, $ax^2 + bx + c$ is negative for all values of x outside the interval $[x_1, x_2]$ and is positive for all values of x within the interval (x_1, x_2). Besides for values of $x = x_1$ or $x = x_2$, the value of the quadratic expression becomes zero (By definition of the root).

Here are a few examples illustrating how quadratic inequalities are solved.

Solve the following inequalities.

Example 1: $x^2 - 5x + 6 > 0$

Solution: (a) The discriminant $D = 25 - 4 \times 6 > 0$ and a is positive (+1); the roots of the quadratic expression are real and distinct: $x_1 = 2$ and $x_2 = 3$.

By the property of quadratic inequalities, we get that the expression is positive outside the interval [2, 3]. Hence, the solution is $x < 2$ and $x > 3$.

We can also see it as $x^2 - 5x + 6 = (x - 2)(x - 3)$ and the given inequality takes the form $(x - 2)(x - 3) > 0$.

The solutions of the inequality are the numbers $x < 2$ (when both factors are negative and their product is positive) and also the numbers $x > 3$ (when both factors are positive and, hence, their product is also positive).

Answer: $x < 2$ and $x > 3$.

Example 2: $2x^2 + x + 1 \geq 0$

Solution: The discriminant $D = 1 - 4 \cdot (-2) = 9 > 0$; the roots of the quadratic expression are real and distinct:

$$x_{1,2} = \frac{-1 \pm \sqrt{9}}{2 \cdot (-2)} = \frac{-1 \pm 3}{-4}$$

hence, $x_1 = -1/2$ and $x_2 = 1$, and consequently,
$-2x^2 + x + 1 = -2(x + 1/2) \times (x - 1)$. We have

$$-2(x + 1/2)(x - 1) \geq 0 \text{ or } (x + 1/2)(x - 1) \leq 0$$

(When dividing both sides of an inequality by a negative number, the sense of the inequality is reversed). The inequality is satisfied by all numbers from the interval

$$[-1/2, 1]$$

Please note that this can also be concluded from the property of quadratic expressions when $D > 0$ and a is negative.

Answer: $-1/2 \leq x \leq 1$.

Example 3: $2x^2 + x - 1 < 0$

Solution: $D = 1 - 4 \cdot(-2) \cdot (-1) < 0$, the coefficient of x^2 is negative. By the property of the quadratic expression when $D < 0$ and a is negative $-2x^2 + x - 1$ attains only negative values.

Answer: x can take any value.

Example 4: $3x^2 - 4x + 5 < 0$

Solution: $D = 16 - 4 \times 3 \times 5 < 0$, the coefficient of x^2 is positive. The quadratic expression $3x^2 - 4x + 5$ takes on only positive values.

Answer: There is no solution.

Example 5: $4x^2 + 4x + 1 > 0$.

Solution: $D = 16 - 4 \times 4 = 0$. The quadratic expression $4x^2 + 4x + 1$ is the square $(2x + 1)^2$, and the given inequality takes the form $(2x + 1)^2 > 0$. It follows that all real numbers x, except for $x = -1/2$, are solutions of the inequality.

Answer: $x \neq -1/2$.

Example 6: Solve the inequality $(a - 2)x^2 - x - 1 \geq 0$

Here, the value of the determinant $D = 1 - 4(-1)(a - 2) = 1 + 4(a - 2) = 4a - 7$

There can then be three cases:

Case 1: $D < 0 \rightarrow a < 7/4$

Then the coefficient of $x^2 \rightarrow a - 2$ is negative.

Hence, the inequality has no solution.

Case 2: $D = 0$

$a = 7/4$. Put $a = 7/4$ in the expression and then the inequality becomes

$-(x - 2)^2 \geq 0$. This can only happen when $x = 2$.

Case 3: $D > 0$

Then $a > 7/4$ and $a \neq 2$, then we find the roots x_1 and x_2 of the quadratic expression:

$$x_1 = \frac{1 + \sqrt{4a - 7}}{2(a - 2)} \quad \text{and} \quad x_2 = \frac{1 - \sqrt{4a - 7}}{2(a - 2)}$$

Using the property of quadratic expression's values for $D > 0$ we get

If $a - 2 < 0$, the quadratic expression takes negative values outside the interval $[x_1, x_2]$. Hence, it will take positive values inside the interval (x_1, x_2).

If $a - 2 > 0$, the quadratic expression takes positive values outside the interval $[x_1, x_2]$ and becomes zero for x_1 and x_2.

If $a - 2 = 0$, then we get a straight linear equation. $-x - 1 \geq 0 \rightarrow x \leq -1$.

System of Inequalities in One Unknown

Let there be given several inequalities in one unknown. If it is required to find the number that will be the solution of all the given equalities, then the set of these inequalities is called a *system of inequalities*.

The solution of a system of inequalities in one unknown is defined as the value of the unknown for which all the inequalities of the system reduce to true numerical inequalities.

To solve a system of inequalities means to find all the solutions of the system or to establish that there is none.

Two systems of inequalities are said to be *equivalent* if any solution of one of them is a solution of the other, and vice versa. If both the systems of inequalities have no solution, then they are also regarded to be equivalent.

Example 1: Solve the system of inequalities:

$$\begin{aligned} 3x - 4 &< 8x + 6 \\ 2x - 1 &> 5x - 4 \\ 11x - 9 &\leq 15x + 3 \end{aligned}$$

Solution: We solve the first inequality:

$$\begin{aligned} 3x - 4 &< 8x + 6 \\ -5x &< 10 \\ x &> -2 \end{aligned}$$

It is fulfilled for $x > -2$.

Then we solve the second inequality

$$\begin{aligned} 2x - 1 &> 5x - 4 \\ -3x &> -3 \\ x &< 1 \end{aligned}$$

It is fulfilled for $x < 1$.

And, finally, we solve the third inequality:

$$\begin{aligned} 11x - 9 &\leq 15x + 3 \\ -4x &\leq 12 \\ x &\geq -3 \end{aligned}$$

It is fulfilled for $x \geq -3$. All the given inequalities are true for $-2 < x < 1$.

Answer: $-2 < x < 1$.

Example 2: Solve the inequality $\frac{2x - 1}{x + 1} < 1$

We have $\frac{2x - 1}{x + 1} - 1 < 0 \rightarrow \frac{x - 2}{x + 1} < 0$

This means that the fraction above has to be negative. A fraction is negative only when the numerator and the denominator have opposite signs.

Hence, the above inequality is equivalent to the following set of 2 inequalities:

$$x - 2 > 0 \quad \text{and} \quad x - 2 < 0$$

and $$x + 1 < 0 \quad\quad x + 1 > 0$$

From the first system of inequalities, we get $x > 2$ or $x < -1$. This cannot happen simultaneously since these are inconsistent.

From the second system of inequalities we get

$$x < 2 \text{ or } x > -1 \text{ i.e } -1 < x < 2$$

Inequalities Containing a Modulus

Result:

$|x| \le a$, where $a > 0$ means the same as the double inequality

$$-a \le x \le a$$

This result is used in solving inequalities containing a modulus.

Example 1: $|2x - 3| \le 5$

This is equivalent to $-5 \le 2x - 3 \le 5$

i.e. $2x - 3 \ge -5$ and $2x - 3 \le 5$

$2x \ge -2$ $\quad x \le 4$

$x \ge -1$

The solution is

$$-1 \le x \le 4$$

Example 2: $|1 - x| > 3$

$|1 - x| = |x - 1|$

Hence, $|x - 1| > 3 \rightarrow x - 1 > 3$ i.e $x > 4$

or $x - 1 < -3$ or $x < -2$

Answer: $x > 4$ or $x < -2$.

Space for Notes

WORKED-OUT PROBLEMS

Problem 14.1 Solve the inequality $\frac{1}{x} < 1$.

Solution $\frac{1}{x} < 1 \Leftrightarrow \frac{1}{x} - 1 < 0 \Leftrightarrow \frac{1-x}{x} < 0 \Leftrightarrow \frac{x-1}{x} > 0$.

This can happen only when both the numerator and denominator take the same sign (Why?)

Case 1: Both are positive: $x - 1 > 0$ and $x > 0$ i.e. $x > 1$.

Case 2: Both are negative: $x - 1 < 0$ and $x < 0$ i.e. $x < 0$.

Answer: **(– ∞, 0) *U* (1 + ∞)**

Problem 14.2 Solve the inequality $\frac{x}{x+2} \le \frac{1}{x}$.

Solution $\frac{x}{x+2} \le \frac{1}{x} \Leftrightarrow \frac{x}{x+2} - \frac{1}{x} \le 0 \Leftrightarrow \frac{x^2 - x - 2}{x(x+2)} \le 0 \Leftrightarrow \frac{(x-2)(x+1)}{x(x+2)} \le 0$.

The function $f(x) = \frac{(x-2)(x+1)}{x(x+2)}$ becomes negative when numerator and denominator are of opposite sign.

Case 1: Numerator positive and denominator negative: This occurs only between $-2 < x < -1$.

Case 2: Numerator negative and Denominator Positive: Numerator is negative when $(x - 2)$ and $(x + 1)$ take opposite signs. This can be got for:

Case A: $x - 2 < 0$ and $x + 1 > 0$ i.e. $x < 2$ and $x > -1$

Case B: $x - 2 > 0$ and $x + 1 < 0$ i.e. $x > 2$ and $x < -1$. Cannot happen.

Hence, the answer is $-2 < x \le 2$.

Problem 14.3 Solve the inequality $\frac{x}{x-3} \le \frac{1}{x}$.

Solution $\frac{x}{x-3} \le \frac{1}{x} \Leftrightarrow \frac{x}{x-3} - \frac{1}{x} \ge 0 \Leftrightarrow \frac{x^2 - x + 3}{x(x-3)} \le 0$. The function $f(x) = \frac{x^2 - x + 3}{x(x-3)}$

The numerator being a quadratic equation with $D < 0$ and $a > 0$, we can see that it will always be positive for all values of x. (From the property of quadratic inequalities).

Further, for the expression to be negative, the denominator should be negative.

That is, $x^2 - 3x < 0$. This will occur when $x < 3$ and x is positive.

Answer: **(0, 3)**

Suppose $F(x) = (x - x_1)^{k1} (x - x_2)^{k2} ... (x_1 - x_n)^{kn}$, where k_1, k_2 …, k_n are integers. If k_j is an even number, then the function $(x - x_j)^{kj}$ does not change sign when x passes through the point x_j and, consequently, the function $F(x)$ does not change sign. If k_p is an odd number, then the function $(x - x_p)^{kp}$ changes sign when x passes through the point x_p and, consequently, the function $F(x)$ also changes sign.

Problem 14.4 Solve the inequality $(x - 1)^2 (x + 1)^3 (x - 4) < 0$.

Solution The above inequality is valid for

Case 1: $x + 1 < 0$ and $x - 4 > 0$

That is, $x < -1$ and $x > 4$ simultaneously. This cannot happen together.

Case 2: $x + 1 > 0$ and $x - 4 < 0$

That is $x > -1$ or $x < 4$, i.e. $-1 < x < 4$ is the answer

Problem 14.5 Solve the inequality $\frac{(x-1)^2 (x+1)^3}{x^4 (x-2)} \le 0$

Solution The above expression becomes negative when the numerator and the denominator take opposite signs.

That means that the numerator is positive and the denominator is negative or vice versa.

Case 1: Numerator positive and denominator negative: The sign of the numerator is determined by the value of $x + 1$ and that of the denominator is determined by $x - 2$.

This condition happens when $x < 2$ and $x > -1$ simultaneously.

Case 2: Numerator negative and denominator positive: This happens when $x < -1$ and $x > 2$ simultaneously. This will never happen.

Hence, the answer is $-1 \le x < 2$.

Space for Rough Work

LEVEL OF DIFFICULTY (I)

Solve the following inequalities:

1. $3x^2 - 7x + 4 \le 0$
   (a) $x > 0$ (b) $x < 0$
   (c) All x (d) None of these
2. $3x^2 - 7x - 6 < 0$
   (a) $-0.66 < x < 3$ (b) $x < -0.66$ or $x > 3$
   (c) $3 < x < 7$ (d) $-2 < x < 2$
3. $3x^2 - 7x + 6 < 0$
   (a) $0.66 < x < 3$ (b) $-0.66 < x < 3$
   (c) $-1 < x < 3$ (d) None of these
4. $x^2 - 3x + 5 > 0$
   (a) $x > 0$ (b) $x < 0$
   (c) Both (a) and (b) (d) $-\infty < x < \infty$
5. $x^2 - 14x - 15 > 0$
   (a) $x < -1$ (b) $15 < x$
   (c) Both (a) and (b) (d) $-1 < x < 15$
6. $2 - x - x^2 \ge 0$
   (a) $-2 \le x \le 1$ (b) $-2 < x < 1$
   (c) $x < -2$ (d) $x > 1$
7. $|x^2 - 4x| < 5$
   (a) $-1 \le x \le 5$ (b) $1 \le x \le 5$
   (c) $-1 \le x \le 1$ (d) $-1 < x < 5$
8. $|x^2 + x| - 5 < 0$
   (a) $x < 0$ (b) $x > 0$
   (c) All values of x (d) None of these
9. $|x^2 - 5x| < 6$
   (a) $-1 < x < 2$ (b) $3 < x < 6$
   (c) Both (a) and (b) (d) $-1 < x < 6$
10. $|x^2 - 2x| < x$
    (a) $1 < x < 3$ (b) $-1 < x < 3$
    (c) $0 < x < 4$ (d) $x > 3$
11. $|x^2 - 2x - 3| < 3x - 3$
    (a) $1 < x < 3$ (b) $-2 < x < 5$
    (c) $x > 5$ (d) $2 < x < 5$
12. $|x^2 - 3x| + x - 2 < 0$
    (a) $(1 - \sqrt{3}) < x < (2 + \sqrt{2})$
    (b) $0 < x < 5$
    (c) $(1 - \sqrt{3}, 2 - \sqrt{2})$
    (d) $1 < x < 4$
13. $x^2 - 7x + 12 < |x - 4|$
    (a) $x < 2$ (b) $x > 4$
    (c) $2 < x < 4$ (d) $2 \le x \le 4$
14. $x^2 - |5x - 3| - x < 2$
    (a) $x > 3 + 2\sqrt{2}$ (b) $x < 3 + 2\sqrt{2}$
    (c) $x > -5$ (d) $-5 < x < 3 + 2\sqrt{2}$
15. $|x - 6| > x^2 - 5x + 9$
    (a) $1 \le x < 3$ (b) $1 < x < 3$
    (c) $2 < x < 5$ (d) $-3 < x < 1$
16. $|x - 6| < x^2 - 5x + 9$
    (a) $x < 1$ (b) $x > 3$
    (c) $1 < x < 3$ (d) Both (a) and (b)
17. $|x - 2| \le 2x^2 - 9x + 9$
    (a) $x > (4 - \sqrt{2})/2$ (b) $x < (5 + \sqrt{3})/2$
    (c) Both (a) and (b) (d) None of these
18. $3x^2 - |x - 3| > 9x - 2$
    (a) $x < (4 - \sqrt{19})/3$ (b) $x > (4 + \sqrt{19})/3$
    (c) Both (a) and (b) (d) $-2 < x < 2$
19. $x^2 - |5x + 8| > 0$
    (a) $x < (5 - \sqrt{57})/2$ (b) $x < (5 + \sqrt{57})/2$
    (c) $x > (5 + \sqrt{57})/2$ (d) Both (a) and (c)
20. $3|x - 1| + x^2 - 7 > 0$
    (a) $x > -1$ (b) $x < -1$
    (c) $x > 2$ (d) Both (b) and (c)
21. $|x - 6| > |x^2 - 5x + 9|$
    (a) $x < 1$ (b) $x > 3$
    (c) $(1 < x < 3)$ (d) both (a) and (b)
22. $(|x - 1| - 3)(|x + 2| - 5) < 0$
    (a) $-7 < x < -2$ and $3 < x < 4$
    (b) $x < -7$ and $x > 4$
    (c) $x < -2$ and $x > 3$
    (d) Any of these
23. $|x^2 - 2x - 8| > 2x$
    (a) $x < 2\sqrt{2}$ (b) $x < 3 + 3\sqrt{5}$
    (c) $x > 2 + 2\sqrt{3}$ (d) Both (a) and (c)
24. $(x - 1)\sqrt{x^2 - x - 2} \ge 0$
    (a) $x \le 2$ (b) $x \ge 2$
    (c) $x \le -2$ (d) $x \ge 0$
25. $(x^2 - 1)\sqrt{x^2 - x - 2} \ge 0$
    (a) $x \le -1$ (b) $x \ge -1$
    (c) $x \ge 2$ (d) (a) and (c)
26. $\sqrt{\frac{x - 2}{1 - 2x}} > -1$
    (a) $0.5 > x$ (b) $x > 2$
    (c) Both (a) and (b) (d) $0.5 < x \le 2$

27. $\sqrt{\dfrac{3x-1}{2-x}} > 1$
(a) $0 < x < 2$ (b) $0.75 < x < 4$
(c) $0.75 < x < 2$ (d) $0 < x < 4$

28. $\sqrt{3x-10} > \sqrt{6-x}$
(a) $4 < x \le 6$ (b) $x < 4$ or $x > 6$
(c) $x < 4$ (d) $x > 8$

29. $\sqrt{x^2-2x-3} < 1$
(a) $(-1-\sqrt{5} < x < -3)$ (b) $1 \le x < (\sqrt{5}-1)$
(c) $x > 1$ (d) None of these

30. $\sqrt{1-\dfrac{x+2}{x^2}} < \dfrac{2}{3}$
(a) $(-6/5) < x \le -1$ or $2 \le x < 3$
(b) $(-6/5) \le x < -1$
(c) $2 \le x < 3$
(d) $(-6/5) \le x < 3$

31. $2\sqrt{x-1} < x$
(a) $x > 1$ (b) $x \ge 1, x \ne 2$
(c) $x < 1$ (d) $1 < x < 5$

32. $\sqrt{x+18} < 2-x$
(a) $x \le -18$ (b) $x < -2$
(c) $x > -2$ (d) $-18 \le x < -2$

33. $x > \sqrt{24+5x}$
(a) $x < 3$ (b) $3 < x \le 4.8$
(c) $x \ge 24/5$ (d) $x > 8$

34. $\sqrt{9x-20} < x$
(a) $4 < x < 5$ (b) $20/9 \le x < 4$
(c) $x > 5$ (d) Both (b) and (c)

35. $\sqrt{x+7} < x$
(a) $x > 2$ (b) $x > \sqrt{30}/2$
(c) $x > (1+\sqrt{29})/2$ (d) $x > 1+\sqrt{29}/2$

36. $\sqrt{2x-1} < x-2$
(a) $x < 5$ (b) $x > 5$
(c) $x > 5$ or $x < -5$ (d) $5 < x < 15$

37. $\sqrt{x+78} < x+6$
(a) $x < 3$ (b) $x > 3$ or $x < 2$
(c) $x > 3$ (d) $3 < x < 10$

38. $\sqrt{5-2x} < 6x-1$
(a) $0.5 < x$ (b) $x < 2.5$
(c) $0.5 < x < 2.5$ (d) $x > 2.5$

39. $\sqrt{x+61} < x+5$
(a) $x < 3$ (b) $x > 3$ or $x < 1$
(c) $x > 3$ (d) $3 < x < 15$

40. $x < \sqrt{2-x}$
(a) $x > 1$ (b) $x < 1$
(c) $-2 < x < 1$ (d) $-1 < x$

41. $x+3 < \sqrt{x+33}$
(a) $x > 3$ (b) $x < 3$
(c) $-3 < x < 3$ (d) $-33 < x < 3$

42. $\sqrt{2x+14} > x+3$
(a) $x < -7$ (b) $-7 \le x < 1$
(c) $x > 1$ (d) $-7 < x < 1$

43. $x-3 < \sqrt{x-2}$
(a) $2 \le x < (7+\sqrt{5})/2$ (b) $2 \le x$
(c) $x < (7+\sqrt{5})/2$ (d) $x \le 2$

44. $x+2 < \sqrt{x+14}$
(a) $-14 \le x < 2$ (b) $x > -14$
(c) $x < 2$ (d) $-11 < x < 2$

45. $x-1 < \sqrt{7-x}$
(a) $x > 3$ (b) $x < 3$
(c) $-53 < x < 3$ (d) $-103 < x < 3$

46. $\sqrt{9x-20} > x$
(a) $x < 4$ (b) $x > 5$
(c) $x \le 4$ or $x \ge 5$ (d) $4 < x < 5$

47. $\sqrt{11-5x} > x-1$
(a) $x > 2, x < 5$ (b) $-3 < x < 2$
(c) $-25 < x < 2$ (d) $x < 2$

48. $\sqrt{x+2} > x$
(a) $-2 \le x < 2$ (b) $-2 \le x$
(c) $x < 2$ (d) $x = -2$ or $x > 2$

Directions for Questions 49 to 53: Find the largest integral x that satisfies the following inequalities.

49. $\dfrac{x-2}{x^2-9} < 0$
(a) $x = -4$ (b) $x = -2$
(c) $x = 3$ (d) None of these

50. $\dfrac{1}{x+1} - \dfrac{2}{x^2-x+1} < \dfrac{1-2x}{x^3+1}$
(a) $x = 1$ (b) $x = 2$
(c) $x = -1$ (d) None of these

51. $\dfrac{x+4}{x^2-9} - \dfrac{2}{x+3} < \dfrac{4x}{3x-x^2}$
(a) $x = 1$ (b) $x = 2$
(c) $x = -1$ (d) None of these

52. $\dfrac{4x+19}{x+5} < \dfrac{4x-17}{x-3}$
(a) $x = 1$ (b) $x = 2$
(c) $x = -1$ (d) None of these

53. $(x+1)(x-3)^2(x-5)(x-4)^2(x-2) < 0$
(a) $x = 1$ (b) $x = -2$
(c) $x = -1$ (d) None of these

Directions for Questions 54 to 69: Solve the following inequalities:

54. $(x-1)(3-x)(x-2)^2 > 0$
(a) $1 < x < 3$ (b) $1 < x < 3$ but $x \neq 2$
(c) $0 < x < 2$ (d) $-1 < x < 3$

55. $\frac{6x-5}{4x+1} < 0$
(a) $-1/4 < x < 1$ (b) $-1/2 < x < 1$
(c) $-1 < x < 1$ (d) $-1/4 < x < 5/6$

56. $\frac{2x-3}{3x-7} > 0$
(a) $x < 3/2$ or $x > 7/3$ (b) $3/2 < x < 7/3$
(c) $x > 7/3$ (d) None of these

57. $\frac{3}{x-2} < 1$
(a) $2 < x < 5$ (b) $x < 2$
(c) $x > 5$ (d) $x < 2$ or $x > 5$

58. $\frac{1}{x-1} \leq 2$
(a) $x < 1$ (b) $x \geq 1.5$
(c) $-5 < x < 1$ (d) Both (a) and (b)
or $x \geq 1.5$

59. $\frac{4x+3}{2x-5} < 6$
(a) $x < 2.5$ (b) $x < 33/8$
(c) $x \geq 2.5$ (d) $x < 2.5$ or $x > 33/8$

60. $\frac{5x-6}{x+6} < 1$
(a) $-6 < x < 6$ (b) $-6 < x < 0$
(c) $-6 < x < 3$ (d) None of these

61. $\frac{5x+8}{4-x} < 2$
(a) $x < 0$ or $x > 4$ (b) $0 < x < 4$
(c) $0 \leq x < 4$ (d) $x > 4$

62. $\frac{x-1}{x+3} > 2$
(a) $x < -7$ (b) $x < -3$
(c) $-7 < x < -3$ (d) None of these

63. $\frac{7x-5}{8x+3} > 4$
(a) $-17/25 < x < -3/8$ (b) $x > -17/25$
(c) $0 < x < 3/8$ (d) $-17/25 < x < 0$

64. $\frac{x}{x-5} > \frac{1}{2}$
(a) $-5 < x < 5$ (b) $-5 < x < 0$
(c) $-5 \leq x \leq 5$ (d) $x < -5$ or $x > 5$

65. $x \leq \frac{6}{x-5}$
(a) $x < -1$ (b) $x > 5$
(c) $x < 6$ (d) $x \leq -1$ or $5 < x \leq 6$

66. $\frac{30x-9}{x-2} \geq 25(x+2)$
(a) $x < -1.4$ or $x > 2$
(b) $x < -1.4$ or $2 < x \leq 2.6$
(c) $x \leq -1.4$ or $2 < x \leq 2.6$
(d) None of these

67. $\frac{4}{x+2} > 3 - x$
(a) $-2 < x < -1$ or $x > 2$
(b) $-2 < x < 2$
(c) $-2 < x < -1$
(d) $0 < x < 3$

68. $x - 17 \geq \frac{60}{x}$
(a) $x < -3$
(b) $x < 20$
(c) $-3 \leq x < 0$ or $x \geq 20$
(d) $-3 < x \leq 0$ or $x \geq 20$

69. $\sqrt{x^2} < x + 1$
(a) $x > 0.5$ (b) $x > 0$
(c) All x (d) $x > -0.5$

70. Find the smallest integral x satisfying the inequality $\frac{x-5}{x^2+5x-14} > 0$
(a) $x = -6$ (b) $x = -3$
(c) $x = -7$ (d) None of these

71. Find the maximum value of x for which $\frac{x+2}{x} \geq x$

Directions for 72 and 73: If $f(x) = |2x - 4|$ and x is an integer. Then answer the following questions:

72. Find the maximum value of x for which $f(x) \leq 5$.
73. Find minimum value of x for which $f(x) \leq 5$.
74. For how many integer values of x, is the expression: $(x-1)(4-x)(x-2)^2 > 0$

Directions for Question numbers 75 and 76:

If $\frac{x^2-5x+6}{|x|+5} \leq 0$

75. Find the minimum value of x, for which the above inequality is true.
76. For how many integer values of x, the above inequality is true.
77. Find the minimum value of x for which $\frac{1}{x-0.5} < 2$ where $x \in I^+$
78. For how many integer values of x is: $\frac{x^2+6x-7}{x^2+1} > 2$
79. Maximum value of x, for which $\frac{x^2-9}{x^2+x+1} \leq 0$

Directions for Question numbers 80 and 81:

$\frac{x^2-7|x|+10}{x^2-8x+16} < 0$

80. For how many negative integral values of x, is the above inequality true?
81. Find the sum of all integer values for which the above inequality is true.

Direction for Question numbers 82 and 83:

$$f(x) = \frac{x^2 - 4x + 5}{x^2 + 7x + 12}$$

82. For how many positive integer values of x is $f(x) \leq 0$
83. For how many negative integer values of x is $f(x) \leq 0$
84. If $f(x) = x^2 + 2|x| + 1$, then for how many real values of x is: $f(x) \leq 0$.
85. If $\frac{1}{|x| - 2} > \frac{1}{3}$ then the least positive integer value of x, for which this inequality is true?

Space for Rough Work

LEVEL OF DIFFICULTY (II)

1. $x^2 - 5|x| + 6 < 0$
(a) $-3 < x < -2$ (b) $2 < x < 3$
(c) Both (a) and (b) (d) $-3 < x < 3$

2. $x^2 - |x| - 2 \geq 0$
(a) $-2 < x < 2$ (b) $x \leq -2$ or $x \geq 2$
(c) $x < -2$ or $x > 1$ (d) $-2 < x < 1$

Directions for Questions 3 to 16: Solve the following polynomial inequalities

3. $(x - 1)(3 - x)(x - 2)^2 > 0$
(a) $1 < x < 2$ (b) $-1 < x < 3$
(c) $-3 < x < -1$ (d) $1 < x < 3, x \neq 2$

4. $\dfrac{0.5}{x - x^2 - 1} < 0$
(a) $x > 0$ (b) $x \leq 0$
(c) $x \geq 0$ (d) For all real x

5. $\dfrac{x^2 - 5x + 6}{x^2 + x + 1} < 0$
(a) $x < 2$ (b) $x > 3$
(c) $2 < x < 3$ (d) $x < 2$ or $x > 3$

6. $\dfrac{x^2 + 2x - 3}{x^2 + 1} < 0$
(a) $x < -3$ (b) $-7 < x < -3$
(c) $-3 < x < 1$ (d) $-7 < x < 1$

7. $\dfrac{(x-1)(x+2)^2}{-1 - x} < 0$
(a) $x < -1$
(b) $x < -1$ or $x > 1$
(c) $x < -1$ and $x \neq 2$
(d) $x < -1$ or $x > 1$ and $x \neq -2$

8. $\dfrac{x^2 + 4x + 4}{2x^2 - x - 1} > 0$
(a) $x < -2$ (b) $x > 1$
(c) $x \neq 2$ (d) None of these

9. $x^4 - 5x^2 + 4 < 0$
(a) $-2 < x < 1$
(b) $-2 < x < 2$
(c) $-2 < x < -1$ or $1 < x < 2$
(d) $1 < x < 2$

10. $x^4 - 2x^2 - 63 \leq 0$
(a) $x \leq -3$ or $x \geq 3$ (b) $-3 \leq x \leq 0$
(c) $0 \leq x \leq 3$ (d) $-3 \leq x \leq 3$

Directions for Questions 11 to 67: Solve the following polynomial and quadratic inequalities

11. $\dfrac{5x - 1}{x^2 + 3} < 1$
(a) $x < 4$ (b) $1 < x < 4$
(c) $x < 1$ or $x > 4$ (d) $1 < x < 3$

12. $\dfrac{x - 2}{x^2 + 1} < -\dfrac{1}{2}$
(a) $-3 < x < 3$ (b) $x < -3$
(c) $-3 < x < 6$ (d) $-3 < x < 1$

13. $\dfrac{x + 1}{(x - 1)^2} < 1$
(a) $x > 3$ or x is negative
(b) $x > 3$
(c) $x > 3$ or $-23 < x < 0$
(d) x is negative and $x > 2$

14. $\dfrac{x^2 - 7x + 12}{2x^2 + 4x + 5} > 0$
(a) $x < 3$ or $x > 4$ (b) $3 < x < 4$
(c) $4 < x < 24$ (d) $0 < x < 3$

15. $\dfrac{x^2 + 6x - 7}{x^2 + 1} \leq 2$
(a) x is negative (b) $x \geq 0$
(c) $x > 0$ or $x < 0$ (d) Always

16. $\dfrac{x^4 + x^2 + 1}{x^2 - 4x - 5} < 0$
(a) $x < -1$ or $x > 5$ (b) $-1 < x < 5$
(c) $x > 5$ (d) $-5 < x < -1$

17. $\dfrac{1 + 3x^2}{2x^2 - 21x + 40} < 0$
(a) $0 < x < 8$ (b) $2.5 < x < 8$
(c) $-8 < x < 8$ (d) $3 < x < 8$

18. $\dfrac{1 + x^2}{x^2 - 5x + 6} < 0$
(a) $x < 2$ (b) $x > 3$
(c) Both a and b (d) $2 < x < 3$

19. $\dfrac{x^4 + x^2 + 1}{x^2 - 4x - 5} > 0$
(a) $-1 < x < 5$ (b) $x < -1$ or $x > 5$
(c) $x \leq -1$ or $x > 5$ (d) $-1 < x < 1$

20. $\dfrac{1 - 2x - 3x^2}{3x - x^2 - 5} > 0$

(a) $x < -1$ or $x > 1/3$ (b) $x < -1$ or $x = 1/3$
(c) $-1 < x < 1/3$ (d) $x < 1/3$

21. $\dfrac{x^2 - 5x + 7}{-2x^2 + 3x + 2} > 0$

(a) $x > 0.5$ (b) $x > -0.5$
(c) $-0.5 < x < 5$ (d) $-0.5 < x < 2$

22. $\dfrac{2x^2 - 3x - 459}{x^2 + 1} > 1$

(a) $x > -20$ (b) $x < 0$
(c) $x < -20$ (d) $-20 < x < 20$

23. $\dfrac{x^2 - 1}{x^2 + x + 1} < 1$

(a) $x > -2$ (b) $x > 2$
(c) $-2 < x < 2$ (d) $x < 2$

24. $\dfrac{1 - 2x - 3x^2}{3x - x^2 - 5} > 0$

(a) $1 < x < 3$ (b) $1 < x < 7$
(c) $-3 < x < 3$ (d) None of these

25. $\dfrac{x}{x^2 - 3x - 4} > 0$

(a) $-1 < x < 0$ (b) $4 < x$
(c) both (a) and (b) (d) $-1 < x < 4$

26. $\dfrac{x^2 + 7x + 10}{x + 2/3} > 0$

(a) $-5 < x < -2$ or $\dfrac{-2}{3} < x < \infty$
(b) $-5 < x < 8$
(c) $x < -2$
(d) $x > -2$

27. $\dfrac{3x^2 - 4x - 6}{2x - 5} < 0$

(a) $x < (2 - \sqrt{22})/3$
(b) $x > (2 + \sqrt{22})/3$
(c) $(2 - \sqrt{22})/3 < x < (2 + \sqrt{22})/3$
(d) None of these

28. $\dfrac{17 - 15x - 2x^2}{x + 3} < 0$

(a) $-8.5 < x \le -3$
(b) $-17 < x < -3$
(c) $-8.5 < x < -3$ or $x > 1$
(d) $-8.5 < x < 1$

29. $\dfrac{x^2 - 9}{3x - x^2 - 24} < 0$

(a) $-3 < x < 3$ (b) $x < -3$ or $x > 3$
(c) $x < -5$ or $x > 5$ (d) $x < -7$ or $x > 7$

30. $\dfrac{x + 7}{x - 5} + \dfrac{3x + 1}{2} \ge 0$

(a) $1 < x < 5$ (b) $-1 < x < 5$
(c) $1 \le x \le 3$ or $x > 5$ (d) $-1 < x < 3$

31. $2x^2 + \dfrac{1}{x} > 0$

(a) $x > 0$ (b) $x < -1/2$
(c) Both (a) and (b) (d) None of these

32. $\dfrac{x^2 - x - 6}{x^2 + 6x} \ge 0$

(a) $x < -6$ (b) $-2 \le x < 0$
(c) $x > 3$ (d) All of these

33. $\dfrac{x^2 - 5x + 6}{x^2 - 11x + 30} < 0$

(a) $x < 3$ or $x > 5$
(b) $2 < x < 4$ or $5 < x < 7$
(c) $2 < x < 3$ or $5 < x < 6$
(d) $2 < x < 3$ or $5 < x < 7$

34. $\dfrac{x^2 - 8x + 7}{4x^2 - 4x + 1} < 0$

(a) $x < 1$ or $x > 7$ (b) $1 < x < 7$
(c) $-7 < x < 1$ (d) $-7 < x < 7$

35. $\dfrac{x^2 - 36}{x^2 - 9x + 18} < 0$

(a) $-6 < x < 3$ (b) $-6 < x < 6$
(c) $x < -6$ or $x > 3$ (d) $-3 < x < 3$

36. $\dfrac{x^2 - 6x + 9}{5 - 4x - x^2} \ge 0$

(a) $-5 < x < 1$ or $x = 3$ (b) $-5 \le x < 1$ or $x = 3$
(c) $-5 < x \le 1$ or $x = 3$ (d) $-5 \le x \le -1$

37. $\dfrac{x - 1}{x + 1} < x$

(a) $x < -1$ (b) $x > -1$
(c) $-1 < x < 1$ (d) For all real values of x

38. $\dfrac{1}{x + 2} < \dfrac{3}{x - 3}$

(a) $-4.5 < x < -2$ (b) $-4.5 < x < -2$ or $3 < x$
(c) $-4.5 < x < -2, x > 3$ (d) (b) or (c)

39. $\dfrac{14x}{x + 1} - \dfrac{9x - 30}{x - 4} < 0$

(a) $-1 < x < 1$ or $4 < x < 6$
(b) $-1 < x < 4, 5 < x < 7$
(c) $1 < x < 4$ or $5 < x < 7$
(d) $-1 < x < 1$ or $5 < x < 7$

40. $\dfrac{5x^2 - 2}{4x^2 - x + 3} < 1$

(a) $x < 1$
(b) $-2 < x < 2$
(c) $-2.7 < x < 1.75$
(d) $(-(1 + \sqrt{21})/2 < x < (\sqrt{21} - 1)/2$

41. $\dfrac{x^2 - 5x + 12}{x^2 - 4x + 5} > 3$

(a) $x > 0.5$ (b) $1/2 < x < 3$
(c) $x < 0.5, x > 3$ (d) $1 < x < 3$

42. $\dfrac{x^2 - 3x + 24}{x^2 - 3x + 3} < 4$

(a) $x < -1$ (b) $4 < x < 8$
(c) $x < 4$ or $x > 8$ (d) None of these

43. $\dfrac{x^2 - 1}{2x + 5} < 3$

(a) $x < -2.5$ or $-2 < x < 8$
(b) $-2.5 < x < -2$
(c) $-2.5 < x < 8$
(d) Both (a) and (b)

44. $\dfrac{x^2 + 1}{4x - 3} > 2$

(a) $x > 7$ (b) $x > 7, x < 87$
(c) $0.75 < x < 1, x > 7$ (d) $0.25 < x < 1, x > 7$

45. $\dfrac{x^2 + 2}{x^2 - 1} < -2$

(a) $-1 < x < 2$
(b) $-1 < x < 1,$
(c) $-1 < x < 0, 0 < x < 1$
(d) $-2 < x < 2$

46. $\dfrac{3x - 5}{x^2 + 4x - 5} > \dfrac{1}{2}$

(a) $x < -5$ (b) $x > 1$
(c) $-5 < x < 1$ (d) $-5 < x < 5$

47. $\dfrac{2x + 3}{x^2 + x - 12} \le \dfrac{1}{2}$

(a) $-4 < x < -3, 3 < x < 6$
(b) $-4 < x < -3, 0 < x < 6$
(c) $x < -4, -3 \le x < 3, x > 6$
(d) $x < -4, x > 6$

48. $\dfrac{5 - 2x}{3x^2 - 2x - 16} < 1$

(a) $x < -\sqrt{7}$ (b) $-2 < x < \sqrt{7}$
(c) $8/3 \le x$ (d) All of these

49. $\dfrac{15 - 4x}{x^2 - x - 12} < 4$

(a) $x < -\sqrt{63}/2, -3 < x < \sqrt{63}/2$
(b) $x > 4$
(c) Both (a) and (b)
(d) $x > 4, x < -63/2$
(e) None of these

50. $\dfrac{1}{x^2 - 5x + 6} > 1/2$

(a) $1 < x < 2, 3 < x < 4$ (b) $1 < x < 4$
(c) $x < 1, x > 3$ (d) None of these

51. $\dfrac{5 - 4x}{3x^2 - x - 4} < 4$

(a) $x < -\dfrac{\sqrt{7}}{2}$ (b) $-1 < x < \dfrac{\sqrt{7}}{2}$
(c) $x > 4/3$ (d) All of these

52. $\dfrac{(x + 2)(x^2 - 2x + 1)}{4 + 3x - x^2} \ge 0$

(a) $x < -2$ or $-1 < x < 4$
(b) $-2 < x < 4$ or $x > 6$
(c) $-2 < x < -1$ or $x > 4$
(d) None of these

53. $\dfrac{4}{1 + x} + \dfrac{2}{1 - x} < 1$

(a) $-1 < x < 1$ (b) $x < -1$
(c) $x > 1$ (d) both (b) and (c)

54. $2 + 3/(x + 1) > 2/x$

(a) $x < -2$ (b) $-1 < x < 0$
(c) $1/2 < x$ (d) All of these

55. $1 + \dfrac{2}{x - 1} > \dfrac{6}{x}$

(a) $0 \le x \le 1$ (b) $2 \le x \le 3$
(c) $-\infty < x < 1$ (d) Always except (a) and (b)

56. $\dfrac{x^4 - 3x^3 + 2x^2}{x^2 - x - 30} > 0$

(a) $x < -5$ (b) $1 < x < 2$
(c) $x > 6$ (d) Both (b) and (c)

57. $\dfrac{x - 1}{x} - \dfrac{x + 1}{x - 1} < 2$

(a) $-1 \le x \le 0$ (b) $1/2 \le x \le 1$
(c) $0 < x < 1/2$ (d) Always except (a) and (b)

58. $\dfrac{2(x - 3)}{x(x - 6)} \le \dfrac{1}{x - 1}$

(a) $x < 0$ (b) $1 < x < 6$
(c) Both (a) and (b) (d) Always except (a) and (b)

59. $\dfrac{2(x - 4)}{(x - 1)(x - 7)} \ge \dfrac{1}{x - 2}$

(a) $1 < x < 2$ or $7 < x$ (b) $2 < x$
(c) $2 < x < 7$ (d) Both (a) and (c)

60. $\frac{2x}{x^2-9} \le \frac{1}{x+2}$

(a) $x < -3$ (b) $-2 < x < 3$
(c) All except (a) and (b) (d) Both (a) and (b)

61. $\frac{1}{x-2} + \frac{1}{x-1} > \frac{1}{x}$

(a) $-\sqrt{2} < x < 0$ or $2 < x$
(b) $\sqrt{2} < x$
(c) $1 < x < \sqrt{2}$
(d) Both (a) and (c)

62. $\frac{7}{(x-2)(x-3)} + \frac{9}{x-3} + 1 < 0$

(a) $-5 < x < 4$
(b) $-5 < x < 1$ and $1 < x < 3$ and x is not 2
(c) $-5 < x < 1$ and $2 < x < 3$
(d) $x < 1$

63. $\frac{20}{(x-3)(x-4)} + \frac{10}{x-4} + 1 > 0$

(a) $x < -2$ (b) $-1 < x < 3$ and $4 < x$
(c) All except (a) and (b) (d) Both (a) and (b)

64. $\frac{(x-2)(x-4)(x-7)}{(x+2)(x+4)(x+7)} > 1$

(a) $x < -7$ (b) $x < -7$ and $-4 < x < -2$
(c) $-4 < x < 2$ (d) None of these

65. $\frac{(x-1)(x-2)(x-3)}{(x+1)(x+2)(x+3)} > 1$

(a) $-3 \le x \le -2$ (b) $x < -3$
(c) $-2 < x < -1$ (d) None of these

66. $(x^2 + 3x + 1)(x^2 + 3x - 3) \ge 5$

(a) $x < -4$ or $-2 < x$
(b) $-2 < x < -1$ or $1 < x$
(c) $x \le -4$; $-2 \le x \le -1$; $1 \le x$
(d) $x < -4$ or $1 < x$

67. $(x^2 - x - 1)(x^2 - x - 7) < -5$

(a) $-2 < x$
(b) $-2 < x < -1$ and $1 < x < 4$
(c) $-2 < x < -1$ and $2 < x < 3$
(d) $-2 < x < 0$ and $2 < x < 3$

Directions for Questions 68 to 92: Solve inequalities based on modulus

68. $|x^3 - 1| \ge 1 - x$

(a) $-1 < x < 0$ (b) $x < -1$
(c) $0 < x$ (d) Always except (a)

69. $\frac{x^2-5x+6}{|x|+7} < 0$

(a) $2 \le x \le 3$ (b) $2 < x$
(c) $1 < x < 3$ (d) $2 < x < 3$

70. $\frac{x^2+6x-7}{|x+4|} < 0$

(a) $-7 < x < -5$ and $-4 < x < 1$
(b) $-7 < x < -5$ and $-4 < x < 0$
(c) $-7 < x < -4$ and $-4 < x < 1$
(d) None of these

71. $\frac{|x-2|}{x-2} > 0$

(a) $2 < x < 10$ (b) $3 \le x$
(c) $2 \le x$ (d) $2 < x$

72. $\left|\frac{2}{x-4}\right| > 1$

(a) $2 < x < 4$; $4 \le x \le 5$ (b) $2 < x \le 4$; $4 \le x \le 5$
(c) $2 < x < 4$; $4 < x < 6$ (d) $2 < x < 4$; $4 \le x \le 6$

73. $\left|\frac{2x-1}{x-1}\right| > 2$

(a) $2 < x$ (b) $1 < x$
(c) $3/4 < x < 1$ (d) Both (b) and (c)

74. $\left|\frac{x^2-3x-1}{x^2+x+1}\right| < 3$

(a) $x < -2$ (b) $-1 < x$
(c) Always except (b) (d) Both (a) and (b)

75. $\frac{x^2-7|x|+10}{x^2-6x+9} < 0$

(a) $-5 \le x \le -2$; $2 < x < 5$
(b) $-5 < x < -2$; $2 < x < 5$
(c) $-5 < x < -2$; $2 < x < 3$; $3 < x < 5$
(d) $-5 < x < -2$; $3 < x < 5$

76. $\frac{|x+3|+x}{x+2} > 1$

(a) $-5 < x \le -2$ (b) $-2 \le x \le -1$
(c) $-1 < x$ (d) Always except (b)

77. $\frac{|x-1|}{x+2} < 1$

(a) $-8 < x \le -3$
(b) $-3 < x \le -2$
(c) Always except $x = -2$
(d) Both (a) and (b)

78. $\frac{|x+2|-x}{x} < 2$

(a) $-5 \le x < 0$ (b) $0 \le x \le 1$
(c) Both (a) and (b) (d) Always except (b)

79. $\frac{1}{|x|-3} < \frac{1}{2}$

(a) $x < -5$ and $-3 < x < 3$
(b) $3 \le x \le 5$

(c) $-5 \le x \le -3$
(d) Always except (b) and (c)

80. $\left|\dfrac{3x}{x^2-4}\right| \le 1$

(a) $x \le -4$ and $-1 \le x \le 1$
(b) $4 \le x$
(c) Both of these
(d) None of these

81. $\left|\dfrac{x^2-5x+4}{x^2-4}\right| \le 1$

(a) $[0 < x < 8/5]\ U\ [5/2 < x < +\infty]$
(b) $[0, 5/2]\ U\ [16/5, +\infty]$
(c) $[0, 8/5]\ U\ [5/2, +\infty]$
(d) $[0, 8/5]\ U\ [5/2, +\infty]$

82. $\dfrac{|x-3|}{x^2-5x+6} \ge 2$

(a) $[3/2, 1]$ (b) $[1, 2]$
(c) $[1.5, 2]$ (d) None of these

83. $\dfrac{x^2-|x|-12}{x-3} \ge 2x$

(a) $-101 < x < 25$ (b) $[-\infty, 3]$
(c) $x \le 3$ (d) $x < 3$

84. $|x| < \dfrac{9}{x}$

(a) $x < -1$ (b) $0 < x < 3$
(c) $1 < x < 3; x < -1$ (d) $-\infty < x < 3$

85. $1 + \dfrac{12}{x^2} < \dfrac{7}{x}$

(a) $x < -2; 2 < x < 3$ (b) $3 \le x < 4$
(c) Both (a) and (b) (d) None of these

86. $\dfrac{(x^2-4x+5)}{(x^2+5x+6)} \ge 0$

(a) $-\infty < x < \infty$ (b) $x < -3$
(c) $x > -2$ (d) Both (b) and (c)

87. $\dfrac{x+1}{x-1} \ge \dfrac{x+5}{x+1}$

(a) $x < -1$ (b) $0 < x < 3$
(c) $1 < x \le 3, x < -1$ (d) $-\infty < x < 3$

88. $\dfrac{x-1}{x^2-x-12} \le 0$

(a) $x < -3; 2 < x < 3$ (b) $3 \le x < 4$
(c) Both (a) and (b) (d) None of these

89. $1 < (3x^2 - 7x + 8)/(x^2 + 1) \le 2$

(a) $1 < x < 6$ (b) $1 \le x < 6$
(c) $1 < x \le 6$ (d) $1 \le x \le 6$

90. If $f'(x) \ge g(x)$, where $f(x) = 5 - 3x + \dfrac{5}{2}x^2 - \dfrac{x^3}{3}$, $g(x) = 3x - 7$

(a) $[2, 3]$ (b) $[2, 4]$
(c) $x = 2.5$ (d) None of these

91. $f'(x) \ge g'(x)$, if $f(x) = 10x^3 - 13x^2 + 7x$, $g(x) = 11x^3 - 15x^2 - 3$

(a) $[-1, 7/3]$ (b) $[-1, 3.5]$
(c) $[-1, 9/3]$ (d) $[1, 7/3]$

92. $\dfrac{1}{x-2} - \dfrac{1}{x} \le \dfrac{2}{x+2}$

(a) $(-2, -1)\ U\ (2, +\infty)$ (b) $-2 < x < 1$
(c) Both (a) and (b) (d) None of these

Directions for Questions 93 to 95: Solve the following irrational inequalities.

93. $(x-1)\sqrt{x^2-x-2} \ge 0$

(a) $x < 2$ (b) $3 \le x < \infty$
(c) Always except (a) (d) Both (a) and (b)

94. $(x^2-1)\sqrt{x^2-x-2} \ge 0$

(a) $x < -1$ (b) $2 \le x$
(c) Both (a) and (b) (d) None of these

95. $\dfrac{\sqrt{x-3}}{x-2} > 0$

(a) $0 \le x < 2$ (b) $x > 3$
(c) $0 < x < 1$ (d) Both (b) and (c)

96. If x satisfies the inequality $|x-1| + |x-2| + |x-3| \ge 6$, then: **IIFT 2010**

(a) $0 \le x \le 4$ (b) $x \le 0$ or $x \ge 4$
(c) $x \le -2$ or $x \ge 3$ (d) $x \ge 3$

Direction 97 and 98: If $f(x) = |x+4| - |x-4|$ and $|f(x)| < 8$, then answer the following questions:

97. How many integer values of x satisfies the above inequality?

(a) 6 (b) 7
(c) 8 (d) 9

98. What is the sum of all positive integer values of x satisfying the above inequality?

(a) 5 (b) 6
(c) 8 (d) 10

Directions for question number 99-100: If $x > -6$ and $\dfrac{1}{x-5} + \dfrac{1}{x+3} < 0$, then answer the following questions:

99. Find the number of positive integer values of x, which satisfy the given inequality.

100. Find the sum of all positive integer values of x which satisfy the given inequality.

Direction for question 101-102:

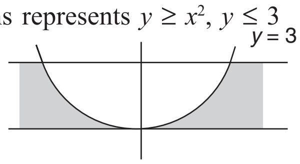
$|4x - 3| \le 8$ and $|3y + 4| \le 17$ then answer the following questions:

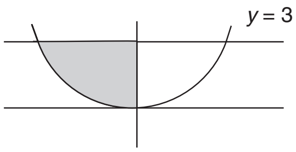
101. Minimum value of $|x|+|y|$ =

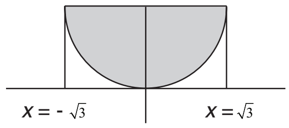
102. Maximum value of $|x| - |y|$ =

103. If $f(x) = \dfrac{x}{2x^2 + 5x + 8}$ for all $x > 0$ what is the greatest value of $f(x)$?
   (a) 1/4 (b) 1/8
   (c) 1/13 (d) 1/5

104. The Shaded portion of which of the following options represents $y \geq x^2$, $y \leq 3$

(a)


(b)


(c)


(d) None of these

105. Find the range of values of x for which $x^4 + 8x < 8x^3 + x^2$
   (a) $(-1,0) \cup (1,8)$ (b) $(0,1) \cup (8,\infty)$
   (c) $(-\infty,-8) \cup (1, 8)$ (d) $(-1, 0) \cup (8, \infty)$

106. If $\dfrac{45}{25x^{14} - 8x^7 + 1} \leq p$
   Then minimum value of p =

107. If $0 \leq x \leq 13$ for how many integer values of x, $7^{x-1} + 11^{x-1} > 170$

108. If $f(x) = \min(3x + 4, 6 - 2x)$ and $f(x) < p$ where p is an integer then the minimum possible value of p = ?

109. For how many non-negative integer values of 'x' is $||||||x - 1| - 2| - 3| - 4| - 5| - 6| - 7| < 9$
   (a) 35 (b) 36
   (c) 37 (d) 38

Space for Rough Work

ANSWER KEY

Level of Difficulty (I)

1. (d)	2. (a)	3. (d)	4. (d)
5. (c)	6. (a)	7. (d)	8. (d)
9. (c)	10. (a)	11. (d)	12. (c)
13. (c)	14. (d)	15. (b)	16. (d)
17. (d)	18. (c)	19. (d)	20. (d)
21. (c)	22. (a)	23. (d)	24. (b)
25. (d)	26. (d)	27. (c)	28. (a)
29. (d)	30. (a)	31. (b)	32. (d)
33. (d)	34. (d)	35. (c)	36. (b)
37. (c)	38. (c)	39. (c)	40. (b)
41. (d)	42. (b)	43. (a)	44. (a)
45. (b)	46. (d)	47. (d)	48. (a)
49. (a)	50. (a)	51. (b)	52. (b)
53. (b)	54. (b)	55. (d)	56. (a)
57. (d)	58. (d)	59. (d)	60. (c)
61. (a)	62. (c)	63. (a)	64. (d)
65. (d)	66. (c)	67. (a)	68. (c)
69. (b)	70. (a)	71. 2	72. 4
73. 0	74. 1	75. 2	76. 2
77. 2	78. 0	79. 3	80. 2
81. –4	82. 5	83. 1	84. 0
85. 3			

Level of Difficulty (II)

1. (c)	2. (b)	3. (d)	4. (d)
5. (c)	6. (c)	7. (d)	8. (d)
9. (c)	10. (d)	11. (c)	12. (d)
13. (a)	14. (a)	15. (d)	16. (b)
17. (b)	18. (d)	19. (b)	20. (a)
21. (d)	22. (c)	23. (a)	24. (d)
25. (c)	26. (a)	27. (d)	28. (c)
29. (b)	30. (c)	31. (d)	32. (d)
33. (c)	34. (b)	35. (a)	36. (a)
37. (b)	38. (b)	39. (a)	40. (d)
41. (b)	42. (d)	43. (a)	44. (c)
45. (c)	46. (c)	47. (c)	48. (d)
49. (c)	50. (a)	51. (d)	52. (a)
53. (d)	54. (d)	55. (d)	56. (d)
57. (d)	58. (c)	59. (a)	60. (d)
61. (d)	62. (b)	63. (d)	64. (b)
65. (d)	66. (c)	67. (c)	68. (d)
69. (d)	70. (c)	71. (d)	72. (c)
73. (d)	74. (d)	75. (c)	76. (d)
77. (c)	78. (d)	79. (d)	80. (c)
81. (a)	82. (d)	83. (d)	84. (b)
85. (d)	86. (d)	87. (c)	88. (d)
89. (d)	90. (d)	91. (a)	92. (d)
93. (c)	94. (c)	95. (b)	96 (b)
97 (b)	98 (b)	99 (3)	100 9
101 0	102 2.75	103 (c)	104 (c)
105 (a)	106 125	107 10	108 6
109 (c)			

Solutions and Shortcuts

While practically solving inequalities remember the following:

1. The answer to an inequality question is always in the form of a range and represents the range of values where the inequality is satisfied.
2. In the cases of all continuous functions, the point at which the range of the correct answer will start, will always be a solution of the same function if written as an equation.

   This rule is only broken for non-continuous functions.

   Hence, if you judge that a function is continuous always check the options for LHS = RHS at the starting point of the option.
3. The correct range has to have two essential properties if it has to be the correct answer:
   (a) The inequality should be satisfied for each and every value of the range.
   (b) There should be no value satisfying the inequality outside the range of the correct option.

   Questions on inequalities are always solved using options and based on (3) (a) and (3) (b) above we would reject an option as the correct answer if:
   (i) we find even a single value not satisfying the inequality within the range of a single option.
   (ii) we can reject given option, even if we find a single value satisfying the inequality but not lying within the range of the option under check.
   I will now show you certain solved questions on this pattern of thinking.

Level of Difficulty (I)

1. At $x = 0$, inequality is not satisfied. Thus, option (c) is rejected. Also $x = 0$ is not a solution of the equation. Since, this is a continuous function, the solution cannot start from 0. Thus options (a) and (b) are not right. Further, we see that the given function is quadratic with real roots. Hence, option (d) is also rejected.
2. At $x = 0$, inequality is satisfied. Hence, options (b) and (c) are rejected. $x = 3$ gives LHS = RHS. and $x = -0.66$ also does the same. Hence. roots of the equation are 3 and – 0.66.

   Thus, option (a) is correct.
3. At $x = 0$, inequality is not satisfied.
   Hence, options (b), (c) are rejected. At $x = 2$, inequality is not satisfied. Hence, option (a) is rejected.

   Thus, option (d) is correct.
4. The given quadratic equation has imaginary roots and is hence always positive.

   Thus, option (d) is correct
5. At $x = 0$ inequality is not satisfied. Thus option (d) is rejected.

$x = -1$ and $x = 15$ are the roots of the quadratic equation. Thus, option (c) is correct.

6. At $x = 0$, inequality is satisfied.
Thus, options, (c) and (d) are rejected.
At $x = 1$, inequality is satisfied
Hence, we choose option (a).
7. At $x = 0$ inequality is satisfied, option (b) is rejected.
At $x = 2$, inequality is satisfied, option (c) is rejected.
At $x = 5$, LHS = RHS.
At $x = -1$, LHS = RHS.
Thus, option (d) is correct.
8. At $x = 0$ inequality is satisfied.
Thus, options (a), (b), are rejected. Option (c) is obviously not true, as there will be values of x at which the inequality would not be satisfied.
Option (d) is correct.
10. At $x = 1$ and $x = 3$ LHS = RHS.
At $x = 2$ inequality is satisfied.
At $x = 0.1$ inequality is not satisfied.
At $x = 2.9$ inequality is satisfied.
At $x = 3.1$ inequality is not satisfied.
Thus, option (a) is correct.
12. The options need to be converted to approximate values before you judge the answer. At $x = 0$, inequality is satisfied.
Thus, option (b) and (d) are rejected.
Option (c) is correct.
13. At $x = 0$, inequality is not satisfied, option (a) is rejected.
At $x = 5$, inequality is not satisfied, option (b) is rejected.
At $x = 2$ inequality is not satisfied.
Option (d) is rejected.
Option (c) is correct.
14. At $x = 0$, inequality is satisfied, option (a) rejected.
At $x = 10$, inequality is not satisfied, option (c) rejected.
At $x = -5$, LHS = RHS.
Also at $x = 5$, inequality is satisfied and at $x = 6$, inequality is not satisfied.
Thus, option (d) is correct.
15. At $x = 2$, inequality is satisfied.
At $x = 0$, inequality is not satisfied.
At $x = 1$, inequality is not satisfied but LHS = RHS.
At $x = 3$, inequality is not satisfied but LHS = RHS.
Thus, option (b) is correct.

Solve other questions of LOD I and LOD II in the same fashion.

71. $\dfrac{x+2}{x} - x \geq 0$

$$\frac{x+2-x^2}{x} \geq 0$$

$$\frac{x^2-x-2}{x} \leq 0$$

$$\frac{(x-2)(x+1)}{x} \leq 0$$

Case 1: $(x - 2)(x + 1) \geq 0$ and $x < 0$
This occurs only for $x \leq -1$
Case 2: $(x - 2)(x + 1) \leq 0$ and x positive.
This occurs when $0 < x \leq 2$
Therefore maximum value of x which satisfies the condition is at $x = 2$.

Solution for 72 & 73:

x is an integer.
$|2x - 4| \leq 5$
$-5 \leq 2x - 4 \leq 5$
$-1 \leq 2x \leq 9$
$-\frac{1}{2} < x \leq \frac{9}{2}$.

72. Maximum value of $x = 4$.
73. Minimum value of $x = 0$.
74. $(x - 1)(4 - x)(x - 2)^2 > 0$
As $(x - 2)^2$ is always non-negative
This means that the product of the first two brackets would be positive. This also means that $(x - 1)(x - 4) < 0$
$\Rightarrow 1 < x < 4$
x has two integer values 2 and 3 between 1 and 4.
But for $x = 2$, $(x - 1)(4 - x)(x - 2)^2 = 0$
$\Rightarrow$ Therefore the given inequality is true only for one integer value of x.

Solution for 75 and 76:

$$\frac{x^2-5x+6}{|x|+5} \leq 0$$

$$\frac{(x-2)(x-3)}{|x|+5} \leq 0$$

Case I: $x > 0$
In this case the expression would become: $\Rightarrow \dfrac{(x-2)(x-3)}{x+5} \leq 0$
$\Rightarrow 2 \leq x \leq 3$
Case II: $x < 0$
In this case the expression would become:

$$\frac{(x-2)(x-3)}{-x+5} \leq 0$$

$$\frac{(x-2)(x-3)}{x-5} \geq 0$$

The above inequality is not satisfied for any negative value of x.

Therefore solution of the above inequality $\rightarrow 2 \le x \le 3$

75. Minimum value of $x = 2$
76. The above inequality is true for two integral values of x.
77. $x \in I^+$ means x is a positive integer.

Now we can check the above inequality by putting positive integer values of x.

For $x = 1, \frac{1}{1-0.5} = 2$, so $x = 1$ is not a solution.

For $x = 2, \frac{1}{2-0.5} = \frac{1}{1.5} < 2$, therefore $x = 2$ is a solution.

The correct answer is $x = 2$.

78. $\frac{x^2+6x-7}{x^2+1} - 2 > 0$

$\frac{-x^2+6x-9}{x^2+1} > 0$

$\frac{(x-3)^2}{x^2+1} < 0$

$(x-3)^2 > 0$, $x^2 + 1 > 0$

So the above inequality is not true for any real value of x.

79. $x^2 + x + 1 > 0$ (It is a quadratic equation with negative discriminant)

Hence, for the expression to be non-positive, the numerator has to be non-positive. *i.e.* $x^2 - 9 \le 0$

$(x-3)(x+3) \le 0$

$\Rightarrow -3 \le x \le 3$

Required maximum value of $x = 3$

Solution for 80 and 81

$\frac{(|x|)^2 - 7|x| + 10}{x^2 - 2\times 4\times x + 4^2} < 0$

$\frac{(|x|-5)(|x|-2)}{(x-4)^2} < 0$

$\Rightarrow 2 < |x| < 5$, but $x \ne 4$

The upper limit defines the range of x as: $-5 < x < 5$

The lower limit defines the range of x as: $x < -2$, $x > 2$

To obey both the limits we will get: $-5 < x < -2$, $2 < x < 5$ & $x \ne 4$.

80. The above inequality is true for $x = -4, -3$.

Therefore for two negative integral values of x the given inequality is true.

81. Required Sum = $-4 - 3 + 3 = -4$

Solution 82 – 83:

82. $\frac{x^2-4x+5}{x^2+7x+12} \le 0$

$\frac{(x-5)(x+1)}{(x+4)(x+3)} \le 0$

$-4 < x < -3$ and $-1 \le x \le 5$

Therefore $f(x) \le 0$ is true for *a* total of 5 positive integer values of x. (1, 2, 3, 4 and 5)

83. $f(x) \le 0$ is true for only one negative integer value of x. (at $x = -1$)
84. $x^2 + 2|x| + 1 \le 0$

$|x|^2 + 2|x| + 1 \le 0$

$(|x| + 1)^2 \le 0$

This is not possible for any real value of x.

85. $\frac{1}{|x|-2} - \frac{1}{3} > 0$

$\frac{(3-|x|+2)}{3(|x|-2)} > 0$

$\frac{5-|x|}{|x|-2} > 0$

$x \in (-5, -2) \cup (2, 5)$

Therefore the least positive integer value of $x = 3$

Level of Difficulty (II)

96. If we put $x = 3$,

Then $|3 - 1| + |3 - 2| + |3 - 3| = 3 \le 6$

Therefore option (*a*), (*c*), (*d*) are not correct.

Hence only option (*b*) is correct.

97. $|f(x)| < 8$

$||x+4| - |x-4|| < 8$

$-8 < |x + 4| - |x - 4| < 8$

Case I : when $x \le -4$

$\Rightarrow |x + 4| - |x - 4| = -x - 4 + x - 4 = -8$, therefore the above inequality is not true for any $x \le -4$

Case II when $x \ge 4$:

$|x + 4| - |x - 4| = x + 4 - x + 4 = 8$, therefore the above inequality is not true for any $x \ge 4$

Case III: when $-4 < x < 4$

$|x + 4| - |x - 4| = x + 4 + x - 4 = 2x$

$\Rightarrow x = \{-3, -2, -1, 0, +1, +2, +3\}$

So the inequality $|f(x)| < 8$ is true for seven integer values of x.

98. Required sum = $1 + 2 + 3 = 6$
99. $\frac{1}{x-5} + \frac{1}{x+3} < 0$

$\frac{x+3+x-5}{(x+3)(x-5)} < 0$

$\frac{2x-2}{(x+3)(x-5)} < 0$

$\frac{x-1}{(x+3)(x-5)} < 0$

$\frac{(x-1)^2}{(x-1)(x+3)(x-5)} < 0$. The numerator of this expression would always be positive.

Hence we need: $(x - 1)(x + 3)(x - 5) < 0$

$x < -3$ or $1 < x < 5$, therefore the given inequality is true for three positive integer values of x i.e x = 2, 3 or 4.

100. Required sum = 2 + 3 + 4= 9

101. $-8 \le 4x - 3 \le 8$

$-5 \le 4x \le 11$

$-\frac{5}{4} \le x \le \frac{11}{4}$

$|3y + 4| \le 17$

$-17 \le 3y + 4 \le 17$

$-7 \le y \le \frac{13}{3}$

x and y both can be zero, therefore minimum value of $|x| + |y| = 0 + 0 = 0$

102. $|x| - |y|$ will be maximum when $|x|$ is maximum and $|y|$ is minimum

$(|x| - |y|)_{max} = |x|_{max} - |y|_{min}$

$= \frac{11}{4} - 0$

$= \frac{11}{4} = 2.75$

103. $f(x) = \frac{x}{2x^2 + 5x + 8} = \frac{1}{2x + \frac{8}{x} + 5}$

$f(x)$ is maximum where denominator is minimum

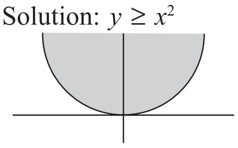
$\frac{\left(2x + \frac{8}{x}\right)}{2} \ge \sqrt{2x \times \frac{8}{x}}$

$2x + \frac{8}{x} \ge 4$

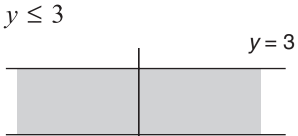
$\left(2x + \frac{8}{x}\right)_{\min} = 8$

$(f(x))_{\max} = \frac{1}{8+5} = \frac{1}{13}$

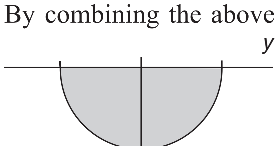
104. Solution: $y \ge x^2$



$y \le 3$



By combining the above graphs we get



Hence option (c) is correct.

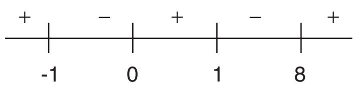
105. $x^4 - 8x^3 - x^2 + 8x < 0$

$x^3(x - 8) - x(x - 8) < 0$

$(x^3 - x)(x - 8) < 0$

$x(x - 1)(x + 1)(x - 8) < 0.$

This expression would be alternately positive and negative in various ranges of x as shown below:



$x \in (-1, 0) \cup (1, 8)$

Hence option (a) is correct.

106. $\frac{45}{(5x^7)^2 - 2 \times 5x^7 \times \frac{4}{5} + \frac{16}{25} - \frac{16}{25} + 1} = \frac{45}{\left(5x^7 - \frac{4}{5}\right)^2 + \frac{9}{25}}$

Minimum value of p is the maximum value of $\frac{45}{25x^{14} - 8x^7 + 1}$

Minimum value of $\left(5x^7 - \frac{4}{5}\right)^2 = 0$

$(p)_{\min} = \frac{45}{\frac{9}{25}} = 5 \times 25 = 125$

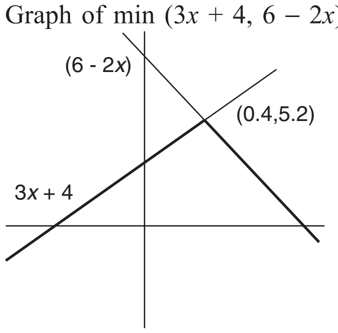
107. For $x = 3$, $7^{3-1} + 11^{3-1} = 7^2 + 11^2 = 170$

$7^{x-1} + 11^{x-1}$ is an increasing function so for $x > 3$, $7^{x-1} + 11^{x-1} > 170$

$\therefore$ $7^{x-1} + 11^{x-1} > 170$ for a total 13 − 3 = 10 values of x.

108. $3x + 4$ is an increasing while $6 - 2x$ is a decreasing function.

Graph of min $(3x + 4, 6 - 2x)$



From the graph it is clear that maximum value of $f(x) = 5.2$, which occurs at $x = 0.4$

$f(x) < p$ (where p is an integer)

Least value of p must be 6.

109. For maximum possible values of x which satisfy the above inequality all the modulus open with a positive sign i.e.

$|x - 1| > 0$, $||x - 1|-2| > 0$, $|||x - 1|-2|-3| > 0$ etc.

Hence for $x = x_{max}$ the given inequality will be

$x - 1 - 2 - 3 - 4 - 5 - 6 - 7 < 9$

$x_{max} < 37$

Hence a total for 37 non-negative integers (0 to 36) are possible. Option (c) is correct.